

# MOMENTUM ANALYSIS OF FLOW SYSTEMS

**W**hen dealing with engineering problems, it is desirable to obtain fast and accurate solutions at minimal cost. Most engineering problems, including those associated with fluid flow, can be analyzed using one of three basic approaches: differential, experimental, and control volume. In *differential approaches*, the problem is formulated accurately using differential quantities, but the solution of the resulting differential equations is difficult, usually requiring the use of numerical methods with extensive computer codes. *Experimental approaches* complemented with dimensional analysis are highly accurate, but they are typically time consuming and expensive. The *finite control volume approach* described in this chapter is remarkably fast and simple and usually gives answers that are sufficiently accurate for most engineering purposes. Therefore, despite the approximations involved, the basic finite control volume analysis performed with paper and pencil has always been an indispensable tool for engineers.

In Chap. 5, the control volume mass and energy analysis of fluid flow systems was presented. In this chapter, we present the finite control volume momentum analysis of fluid flow problems. First we give an overview of Newton's laws and the conservation relations for linear and angular momentum. Then using the Reynolds transport theorem, we develop the linear momentum and angular momentum equations for control volumes and use them to determine the forces and torques associated with fluid flow.

## OBJECTIVES

When you finish reading this chapter, you should be able to

- Identify the various kinds of forces and moments acting on a control volume
- Use control volume analysis to determine the forces associated with fluid flow
- Use control volume analysis to determine the moments caused by fluid flow and the torque transmitted

Steady swimming of the jellyfish *Aurelia aurita*. Fluorescent dye placed directly upstream of the animal is drawn underneath the bell as the body relaxes and forms vortex rings below the animal as the body contracts and ejects fluid. The vortex rings simultaneously induce flows for both feeding and propulsion.

*Adapted from Dabiri et al., J. Exp. Biol. 208: 1257–1265.  
Photo credit: Sean P. Colin and John H. Costello.*



## 6-1 ■ NEWTON'S LAWS

Newton's laws are relations between motions of bodies and the forces acting on them. Newton's first law states that *a body at rest remains at rest, and a body in motion remains in motion at the same velocity in a straight path when the net force acting on it is zero*. Therefore, a body tends to preserve its state of inertia. Newton's second law states that *the acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass*. Newton's third law states that *when a body exerts a force on a second body, the second body exerts an equal and opposite force on the first*. Therefore, the direction of an exposed reaction force depends on the body taken as the system.

For a rigid body of mass  $m$ , Newton's second law is expressed as

$$\text{Newton's second law:} \quad \vec{F} = m\vec{a} = m \frac{d\vec{V}}{dt} = \frac{d(m\vec{V})}{dt} \quad (6-1)$$

where  $\vec{F}$  is the net force acting on the body and  $\vec{a}$  is the acceleration of the body under the influence of  $\vec{F}$ .

The product of the mass and the velocity of a body is called the *linear momentum* or just the *momentum* of the body. The momentum of a rigid body of mass  $m$  moving with velocity  $\vec{V}$  is  $m\vec{V}$  (Fig. 6-1). Then Newton's second law expressed in Eq. 6-1 can also be stated as *the rate of change of the momentum of a body is equal to the net force acting on the body* (Fig. 6-2). This statement is more in line with Newton's original statement of the second law, and it is more appropriate for use in fluid mechanics when studying the forces generated as a result of velocity changes of fluid streams. Therefore, in fluid mechanics, Newton's second law is usually referred to as the *linear momentum equation*.

The momentum of a system remains constant only when the net force acting on it is zero, and thus the momentum of such a system is conserved. This is known as the *conservation of momentum principle*. This principle has proven to be a very useful tool when analyzing collisions such as those between balls; between balls and rackets, bats, or clubs; and between atoms or subatomic particles; and explosions such as those that occur in rockets, missiles, and guns. In fluid mechanics, however, the net force acting on a system is typically *not* zero, and we prefer to work with the linear momentum equation rather than the conservation of momentum principle.

Note that force, acceleration, velocity, and momentum are vector quantities, and as such they have direction as well as magnitude. Also, momentum is a constant multiple of velocity, and thus the direction of momentum is the direction of velocity as shown in Fig 6-1. Any vector equation can be written in scalar form for a specified direction using magnitudes, e.g.,  $F_x = ma_x = d(mV_x)/dt$  in the  $x$ -direction.

The counterpart of Newton's second law for rotating rigid bodies is expressed as  $\vec{M} = I\vec{\alpha}$ , where  $\vec{M}$  is the net moment or torque applied on the body,  $I$  is the moment of inertia of the body about the axis of rotation, and  $\vec{\alpha}$  is the angular acceleration. It can also be expressed in terms of the rate of change of angular momentum  $d\vec{H}/dt$  as

$$\text{Angular momentum equation:} \quad \vec{M} = I\vec{\alpha} = I \frac{d\vec{\omega}}{dt} = \frac{d(I\vec{\omega})}{dt} = \frac{d\vec{H}}{dt} \quad (6-2)$$

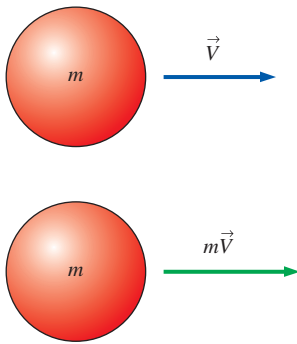


FIGURE 6-1

Linear momentum is the product of mass and velocity, and its direction is the direction of velocity.

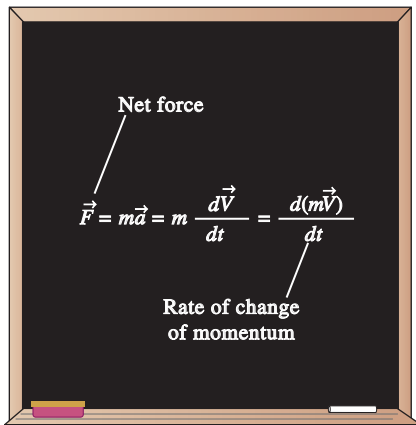


FIGURE 6-2

Newton's second law is also expressed as *the rate of change of the momentum of a body is equal to the net force acting on it*.

where  $\vec{\omega}$  is the angular velocity. For a rigid body rotating about a fixed  $x$ -axis, the angular momentum equation is written in scalar form as

$$\text{Angular momentum about } x\text{-axis:} \quad M_x = I_x \frac{d\omega_x}{dt} = \frac{dH_x}{dt} \quad (6-3)$$

The angular momentum equation can be stated as *the rate of change of the angular momentum of a body is equal to the net torque acting on it* (Fig. 6-3).

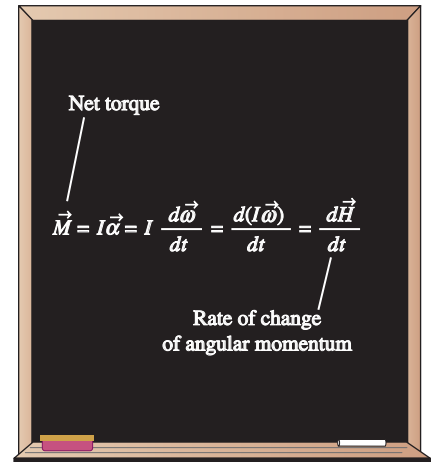
The total angular momentum of a rotating body remains constant when the net torque acting on it is zero, and thus the angular momentum of such systems is conserved. This is known as the *conservation of angular momentum principle* and is expressed as  $I\omega = \text{constant}$ . Many interesting phenomena such as ice skaters spinning faster when they bring their arms close to their bodies and divers rotating faster when they curl after the jump can be explained easily with the help of the conservation of angular momentum principle (in both cases, the moment of inertia  $I$  is decreased and thus the angular velocity  $\omega$  is increased as the outer parts of the body are brought closer to the axis of rotation).

## 6-2 ■ CHOOSING A CONTROL VOLUME

We now briefly discuss how to *wisely* select a control volume. A control volume can be selected as any arbitrary region in space through which fluid flows, and its bounding control surface can be fixed, moving, and even deforming during flow. The application of a basic conservation law is a systematic procedure for bookkeeping or accounting of the quantity under consideration, and thus it is extremely important that the boundaries of the control volume are well defined during an analysis. Also, the flow rate of any quantity into or out of a control volume depends on the flow velocity *relative to the control surface*, and thus it is essential to know if the control volume remains at rest during flow or if it moves.

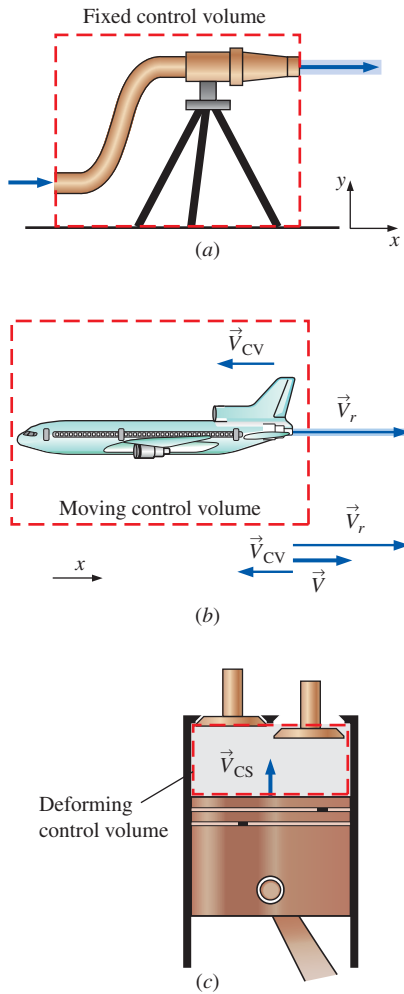
Many flow systems involve stationary hardware firmly fixed to a stationary surface, and such systems are best analyzed using *fixed* control volumes. When determining the reaction force acting on a tripod holding the nozzle of a hose, for example, a natural choice for the control volume is one that passes perpendicularly through the nozzle exit flow and through the bottom of the tripod legs (Fig. 6-4a). This is a fixed control volume, and the water velocity relative to a fixed point on the ground is the same as the water velocity relative to the nozzle exit plane.

When analyzing flow systems that are moving or deforming, it is usually more convenient to allow the control volume to *move* or *deform*. When determining the thrust developed by the jet engine of an airplane cruising at constant velocity, for example, a wise choice of control volume is one that encloses the airplane and cuts through the nozzle exit plane (Fig. 6-4b). The control volume in this case moves with velocity  $\vec{V}_{CV}$ , which is identical to the cruising velocity of the airplane relative to a fixed point on earth. When determining the flow rate of exhaust gases leaving the nozzle, the proper velocity to use is the velocity of the exhaust gases relative to the nozzle exit plane, that is, the *relative velocity*  $\vec{V}_r$ . Since the entire control volume moves at velocity  $\vec{V}_{CV}$ , the relative velocity becomes  $\vec{V}_r = \vec{V} - \vec{V}_{CV}$ , where  $\vec{V}$  is the *absolute velocity* of the exhaust gases, i.e., the velocity relative to a fixed



**FIGURE 6-3**

The rate of change of the angular momentum of a body is equal to the net torque acting on it.

**FIGURE 6-4**

Examples of (a) fixed, (b) moving, and (c) deforming control volumes.

point on earth. Note that  $\vec{V}_r$  is the fluid velocity expressed relative to a coordinate system moving *with* the control volume. Also, this is a vector equation, and velocities in opposite directions have opposite signs. For example, if the airplane is cruising at 500 km/h to the left, and the velocity of the exhaust gases is 800 km/h to the right relative to the ground, the velocity of the exhaust gases relative to the nozzle exit is

$$\vec{V}_r = \vec{V} - \vec{V}_{cv} = 800\vec{i} - (-500\vec{i}) = 1300\vec{i} \text{ km/h}$$

That is, the exhaust gases leave the nozzle at 1300 km/h to the right relative to the nozzle exit (in the direction opposite to that of the airplane); this is the velocity that should be used when evaluating the outflow of exhaust gases through the control surface (Fig. 6-4b). Note that the exhaust gases would appear motionless to an observer on the ground if the relative velocity were equal in magnitude to the airplane velocity.

When analyzing the purging of exhaust gases from a reciprocating internal combustion engine, a wise choice for the control volume is one that comprises the space between the top of the piston and the cylinder head (Fig. 6-4c). This is a *deforming* control volume, since part of the control surface moves relative to other parts. The relative velocity for an inlet or outlet on the deforming part of a control surface (there are no such inlets or outlets in Fig. 6-4c) is then given by  $\vec{V}_r = \vec{V} - \vec{V}_{cs}$  where  $\vec{V}$  is the absolute fluid velocity and  $\vec{V}_{cs}$  is the control surface velocity, both relative to a fixed point outside the control volume. Note that  $\vec{V}_{cs} = \vec{V}_{cv}$  for moving but nondeforming control volumes, and  $\vec{V}_{cs} = \vec{V}_{cv} = 0$  for fixed ones.

### 6-3 ■ FORCES ACTING ON A CONTROL VOLUME

The forces acting on a control volume consist of **body forces** that act throughout the entire body of the control volume (such as gravity, electric, and magnetic forces) and **surface forces** that act on the control surface (such as pressure and viscous forces and reaction forces at points of contact). Only external forces are considered in the analysis. Internal forces (such as the pressure force between a fluid and the inner surfaces of the flow section) are not considered in a control volume analysis unless they are exposed by passing the control surface through that area.

In control volume analysis, the sum of all forces acting on the control volume at a particular instant in time is represented by  $\Sigma \vec{F}$  and is expressed as

$$\text{Total force acting on control volume:} \quad \Sigma \vec{F} = \Sigma \vec{F}_{\text{body}} + \Sigma \vec{F}_{\text{surface}} \quad (6-4)$$

Body forces act on each volumetric portion of the control volume. The body force acting on a differential element of fluid of volume  $dV$  within the control volume is shown in Fig. 6-5, and we must perform a volume integral to account for the net body force on the entire control volume. *Surface forces* act on each portion of the control surface. A differential surface element of area  $dA$  and unit outward normal  $\vec{n}$  on the control surface is shown in Fig. 6-5, along with the surface force acting on it. We must perform an area integral to obtain the net surface force acting on the entire control surface. As sketched, the surface force may act in a direction independent of that of the outward normal vector.

The most common body force is that of **gravity**, which exerts a downward force on every differential element of the control volume. While other body forces, such as electric and magnetic forces, may be important in some analyses, we consider only gravitational forces here.

The differential body force  $d\vec{F}_{\text{body}} = d\vec{F}_{\text{gravity}}$  acting on the small fluid element shown in Fig. 6–6 is simply its weight,

Gravitational force acting on a fluid element: 
$$d\vec{F}_{\text{gravity}} = \rho \vec{g} dV \quad (6-5)$$

where  $\rho$  is the average density of the element and  $\vec{g}$  is the gravitational vector. In Cartesian coordinates we adopt the convention that  $\vec{g}$  acts in the negative  $z$ -direction, as in Fig. 6–6, so that

Gravitational vector in Cartesian coordinates: 
$$\vec{g} = -g\vec{k} \quad (6-6)$$

Note that the coordinate axes in Fig. 6–6 are oriented so that the gravity vector acts *downward* in the  $-z$ -direction. On earth at sea level, the gravitational constant  $g$  is equal to  $9.807 \text{ m/s}^2$ . Since gravity is the only body force being considered, integration of Eq. 6–5 yields

Total body force acting on control volume: 
$$\sum \vec{F}_{\text{body}} = \int_{\text{CV}} \rho \vec{g} dV = m_{\text{CV}} \vec{g} \quad (6-7)$$

Surface forces are not as simple to analyze since they consist of both *normal* and *tangential* components. Furthermore, while the physical force acting on a surface is independent of orientation of the coordinate axes, the *description* of the force in terms of its coordinate components changes with orientation (Fig. 6–7). In addition, we are rarely fortunate enough to have each of the control surfaces aligned with one of the coordinate axes. While not desiring to delve too deeply into tensor algebra, we are forced to define a **second-order tensor** called the **stress tensor**  $\sigma_{ij}$  in order to adequately describe the surface stresses at a point in the flow,

Stress tensor in Cartesian coordinates: 
$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \quad (6-8)$$

The diagonal components of the stress tensor,  $\sigma_{xx}$ ,  $\sigma_{yy}$ , and  $\sigma_{zz}$ , are called **normal stresses**; they are composed of pressure (which always acts inwardly normal) and viscous stresses. Viscous stresses are discussed in more detail in Chap. 9. The off-diagonal components,  $\sigma_{xy}$ ,  $\sigma_{zx}$ , etc., are called **shear stresses**; since pressure can act only normal to a surface, shear stresses are composed entirely of viscous stresses.

When the face is not parallel to one of the coordinate axes, mathematical laws for axes rotation and tensors can be used to calculate the normal and tangential components acting at the face. In addition, an alternate notation called **tensor notation** is convenient when working with tensors but is usually reserved for graduate studies. (For a more in-depth analysis of tensors and tensor notation see, for example, Kundu and Cohen, 2011.)

In Eq. 6–8,  $\sigma_{ij}$  is defined as the stress (force per unit area) in the  $j$ -direction acting on a face whose normal is in the  $i$ -direction. Note that  $i$  and  $j$  are merely *indices* of the tensor and are not the same as unit vectors  $\vec{i}$  and  $\vec{j}$ . For example,  $\sigma_{xy}$  is defined as positive for the stress pointing in the  $y$ -direction on a face whose outward normal is in the  $x$ -direction. This component of the

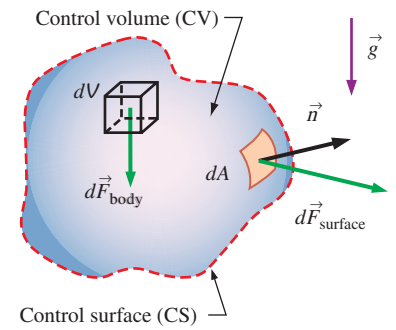


FIGURE 6–5

The total force acting on a control volume is composed of body forces and surface forces; body force is shown on a differential volume element, and surface force is shown on a differential surface element.

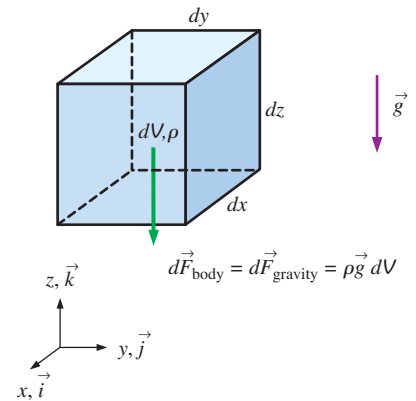
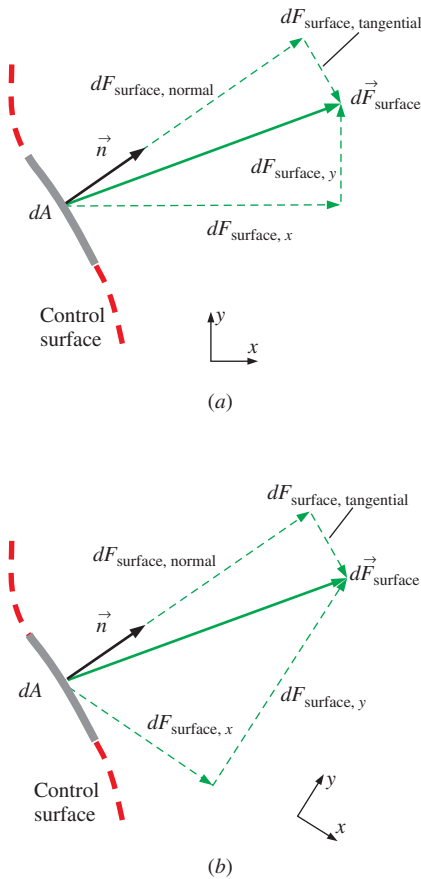


FIGURE 6–6

The gravitational force acting on a differential volume element of fluid is equal to its weight; the axes are oriented so that the gravity vector acts *downward* in the negative  $z$ -direction.



**FIGURE 6-7**

When coordinate axes are rotated (a) to (b), the components of the surface force change, even though the force itself remains the same; only two dimensions are shown here.

stress tensor, along with the other eight components, is shown in Fig. 6–8 for the case of a differential fluid element aligned with the axes in Cartesian coordinates. All the components in Fig. 6–8 are shown on positive faces (right, top, and front) and in their positive orientation by definition. Positive stress components on the *opposing* faces of the fluid element (not shown) point in exactly opposite directions.

The dot product of a second-order tensor and a vector yields a second vector; this operation is often called the **contracted product** or the **inner product** of a tensor and a vector. In our case, it turns out that the inner product of the stress tensor  $\sigma_{ij}$  and the unit outward normal vector  $\vec{n}$  of a differential surface element yields a vector whose magnitude is the force per unit area acting on the surface element and whose direction is the direction of the surface force itself. Mathematically we write

$$\text{Surface force acting on a differential surface element: } d\vec{F}_{\text{surface}} = \sigma_{ij} \cdot \vec{n} dA \quad (6-9)$$

Finally, we integrate Eq. 6–9 over the entire control surface,

$$\text{Total surface force acting on control surface: } \sum \vec{F}_{\text{surface}} = \int_{\text{CS}} \sigma_{ij} \cdot \vec{n} dA \quad (6-10)$$

Substitution of Eqs. 6–7 and 6–10 into Eq. 6–4 yields

$$\sum \vec{F} = \sum \vec{F}_{\text{body}} + \sum \vec{F}_{\text{surface}} = \int_{\text{CV}} \rho \vec{g} dV + \int_{\text{CS}} \sigma_{ij} \cdot \vec{n} dA \quad (6-11)$$

This equation turns out to be quite useful in the derivation of the differential form of conservation of linear momentum, as discussed in Chap. 9. For practical control volume analysis, however, it is rare that we need to use Eq. 6–11, especially the cumbersome surface integral that it contains.

A careful selection of the control volume enables us to write the total force acting on the control volume,  $\sum \vec{F}$ , as the sum of more readily available quantities like weight, pressure, and reaction forces. We recommend the following for control volume analysis:

$$\text{Total force: } \underbrace{\sum \vec{F}}_{\text{total force}} = \underbrace{\sum \vec{F}_{\text{gravity}}}_{\text{body force}} + \underbrace{\sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}}}_{\text{surface forces}} \quad (6-12)$$

The first term on the right-hand side of Eq. 6–12 is the body force *weight*, since gravity is the only body force we are considering. The other three terms combine to form the net surface force; they are pressure forces, viscous forces, and “other” forces acting on the control surface.  $\sum \vec{F}_{\text{other}}$  is composed of reaction forces required to turn the flow; forces at bolts, cables, struts, or walls through which the control surface cuts; etc.

All these surface forces arise as the control volume is isolated from its surroundings for analysis, and the effect of any detached object is accounted for by a force at that location. This is similar to drawing a free-body diagram in your statics and dynamics classes. We should choose the control volume such that forces that are not of interest remain internal, and thus they do not complicate the analysis. A well-chosen control volume exposes only the forces that are to be determined (such as reaction forces) and a minimum number of other forces.

A common simplification in the application of Newton's laws of motion is to subtract the *atmospheric pressure* and work with gage pressures. This is because atmospheric pressure acts in all directions, and its effect cancels out in every direction (Fig. 6–9). This means we can also ignore the pressure forces at outlet sections where the fluid is discharged at subsonic velocities to the atmosphere since the discharge pressures in such cases are very near atmospheric pressure.

As an example of how to wisely choose a control volume, consider control volume analysis of water flowing steadily through a faucet with a partially closed gate valve spigot (Fig. 6–10). It is desired to calculate the net force on the flange to ensure that the flange bolts are strong enough. There are many possible choices for the control volume. Some engineers restrict their control volumes to the fluid itself, as indicated by CV A (the purple control volume) in Fig. 6–10. With this control volume, there are pressure forces that vary along the control surface, there are viscous forces along the pipe wall and at locations inside the valve, and there is a body force, namely, the weight of the water in the control volume. Fortunately, to calculate the net force on the flange, we do *not* need to integrate the pressure and viscous stresses all along the control surface. Instead, we can lump the unknown pressure and viscous forces together into one reaction force, representing the net force of the walls on the water. This force, plus the weight of the faucet and the water, is equal to the net force on the flange. (We must be very careful with our signs, of course.)

When choosing a control volume, you are not limited to the fluid alone. Often it is more convenient to slice the control surface *through* solid objects such as walls, struts, or bolts as illustrated by CV B (the red control volume) in Fig. 6–10. A control volume may even surround an entire object, like the one shown here. Control volume B is a wise choice because we are not concerned with any details of the flow or even the geometry inside the control volume. For the case of CV B, we assign a net reaction force acting at the portions of the control surface that slice through the flange bolts. Then, the only other things we need to know are the gage pressure of the water at the flange (the inlet to the control volume) and the weights of the water and the faucet assembly. The pressure everywhere else along the control surface is atmospheric (zero gage pressure) and cancels out. This problem is revisited in Section 6–4, Example 6–7.

## 6–4 ■ THE LINEAR MOMENTUM EQUATION

Newton's second law for a system of mass  $m$  subjected to net force  $\Sigma \vec{F}$  is expressed as

$$\Sigma \vec{F} = m\vec{a} = m \frac{d\vec{V}}{dt} = \frac{d}{dt}(m\vec{V}) \quad (6-13)$$

where  $m\vec{V}$  is the **linear momentum** of the system. Noting that both the density and velocity may change from point to point within the system, Newton's second law can be expressed more generally as

$$\Sigma \vec{F} = \frac{d}{dt} \int_{\text{sys}} \rho \vec{V} dV \quad (6-14)$$

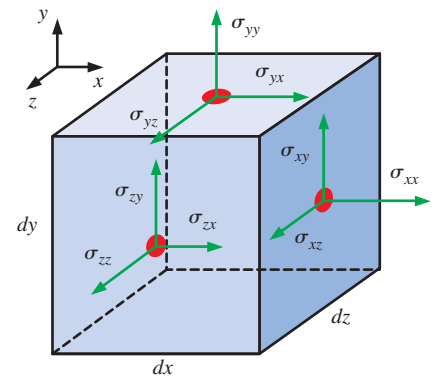


FIGURE 6–8

Components of the stress tensor in Cartesian coordinates on the right, top, and front faces.

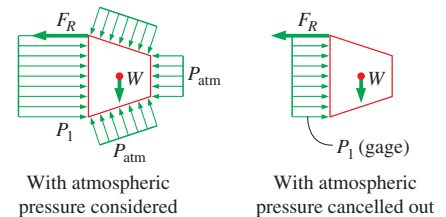


FIGURE 6–9

Atmospheric pressure acts in all directions, and thus it can be ignored when performing force balances since its effect cancels out in every direction.

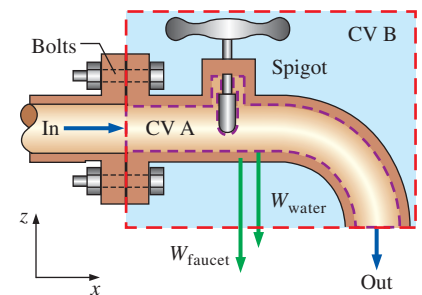


FIGURE 6–10

Cross section through a faucet assembly, illustrating the importance of choosing a control volume wisely; CV B is much easier to work with than CV A.

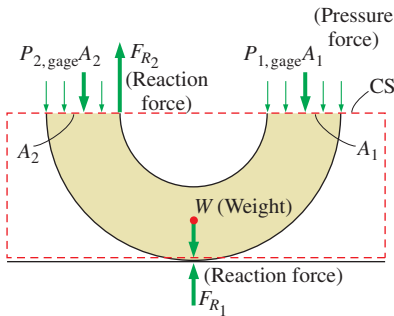
$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt} \int_{\text{CV}} \rho b \, dV + \int_{\text{CS}} \rho b (\vec{V}_r \cdot \vec{n}) \, dA$$

$$B = m\vec{V} \quad b = \vec{V} \quad b = \vec{V}$$

$$\frac{d(m\vec{V})_{\text{sys}}}{dt} = \frac{d}{dt} \int_{\text{CV}} \rho \vec{V} \, dV + \int_{\text{CS}} \rho \vec{V} (\vec{V}_r \cdot \vec{n}) \, dA$$

FIGURE 6–11

The linear momentum equation is obtained by replacing  $B$  in the Reynolds transport theorem by the momentum  $m\vec{V}$ , and  $b$  by the momentum per unit mass  $\vec{V}$ .



An 180° elbow supported by the ground

FIGURE 6–12

In most flow systems, the sum of forces  $\Sigma \vec{F}$  consists of weights, pressure forces, and reaction forces. Gage pressures are used here since atmospheric pressure cancels out on all sides of the control surface.

where  $\rho \vec{V} \, dV$  is the momentum of a differential element  $dV$ , which has mass  $\delta m = \rho \, dV$ . Therefore, Newton's second law can be stated as *the sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system*. This statement is valid for a coordinate system that is at rest or moves with a constant velocity, called an *inertial coordinate system* or *inertial reference frame*. Accelerating systems such as aircraft during takeoff are best analyzed using noninertial (or accelerating) coordinate systems fixed to the aircraft. Note that Eq. 6–14 is a vector relation, and thus the quantities  $\vec{F}$  and  $\vec{V}$  have direction as well as magnitude.

Equation 6–14 is for a given mass of a solid or fluid and is of limited use in fluid mechanics since most flow systems are analyzed using control volumes. The *Reynolds transport theorem* developed in Section 4–6 provides the necessary tools to shift from the system formulation to the control volume formulation. Setting  $b = \vec{V}$  and thus  $B = m\vec{V}$ , the Reynolds transport theorem is expressed for linear momentum as (Fig. 6–11)

$$\frac{d(m\vec{V})_{\text{sys}}}{dt} = \frac{d}{dt} \int_{\text{CV}} \rho \vec{V} \, dV + \int_{\text{CS}} \rho \vec{V} (\vec{V}_r \cdot \vec{n}) \, dA \quad (6-15)$$

The left-hand side of this equation is, from Eq. 6–13, equal to  $\Sigma \vec{F}$ . Substituting, the general form of the linear momentum equation that applies to fixed, moving, or deforming control volumes is

$$\text{General:} \quad \Sigma \vec{F} = \frac{d}{dt} \int_{\text{CV}} \rho \vec{V} \, dV + \int_{\text{CS}} \rho \vec{V} (\vec{V}_r \cdot \vec{n}) \, dA \quad (6-16)$$

which is stated in words as

$$\left( \begin{array}{l} \text{The sum of all} \\ \text{external forces} \\ \text{acting on a CV} \end{array} \right) = \left( \begin{array}{l} \text{The time rate of change} \\ \text{of the linear momentum} \\ \text{of the contents of the CV} \end{array} \right) + \left( \begin{array}{l} \text{The net flow rate of} \\ \text{linear momentum out of the} \\ \text{control surface by mass flow} \end{array} \right)$$

Here  $\vec{V}_r = \vec{V} - \vec{V}_{\text{CS}}$  is the fluid velocity relative to the control surface (for use in mass flow rate calculations at all locations where the fluid crosses the control surface), and  $\vec{V}$  is the fluid velocity as viewed from an inertial reference frame. The product  $\rho (\vec{V}_r \cdot \vec{n}) \, dA$  represents the mass flow rate through area element  $dA$  into or out of the control volume.

For a fixed control volume (no motion or deformation of the control volume),  $\vec{V}_r = \vec{V}$  and the linear momentum equation becomes

$$\text{Fixed CV:} \quad \Sigma \vec{F} = \frac{d}{dt} \int_{\text{CV}} \rho \vec{V} \, dV + \int_{\text{CS}} \rho \vec{V} (\vec{V} \cdot \vec{n}) \, dA \quad (6-17)$$

Note that the momentum equation is a *vector equation*, and thus each term should be treated as a vector. Also, the components of this equation can be resolved along orthogonal coordinates (such as  $x$ ,  $y$ , and  $z$  in the Cartesian coordinate system) for convenience. The sum of forces  $\Sigma \vec{F}$  in most cases consists of weights, pressure forces, and reaction forces (Fig. 6–12). The momentum equation is commonly used to calculate the forces (usually on support systems or connectors) induced by the flow.



## Special Cases

Most momentum problems considered in this text are steady. During *steady flow*, the amount of momentum within the control volume remains constant, and thus the time rate of change of linear momentum of the contents of the control volume (the second term of Eq. 6-16) is zero. Thus,

Steady flow: 
$$\sum \vec{F} = \int_{CS} \rho \vec{V} (\vec{V}_r \cdot \vec{n}) dA \quad (6-18)$$

For a case in which a non-deforming control volume moves at constant velocity (an inertial reference frame), the *first*  $\vec{V}$  in Eq. 6-18 may also be taken relative to the moving control surface.

While Eq. 6-17 is exact for fixed control volumes, it is not always convenient when solving practical engineering problems because of the integrals. Instead, as we did for conservation of mass, we would like to rewrite Eq. 6-17 in terms of average velocities and mass flow rates through inlets and outlets. In other words, our desire is to rewrite the equation in *algebraic* rather than *integral* form. In many practical applications, fluid crosses the boundaries of the control volume at one or more inlets and one or more outlets, and carries with it some momentum into or out of the control volume. For simplicity, we always draw our control surface such that it slices normal to the inflow or outflow velocity at each such inlet or outlet (Fig. 6-13).

The mass flow rate  $\dot{m}$  into or out of the control volume across an inlet or outlet at which  $\rho$  is nearly constant is

Mass flow rate across an inlet or outlet: 
$$\dot{m} = \int_{A_c} \rho (\vec{V} \cdot \vec{n}) dA_c = \rho V_{avg} A_c \quad (6-19)$$

Comparing Eq. 6-19 to Eq. 6-17, we notice an extra velocity in the control surface integral of Eq. 6-17. If  $\vec{V}$  were uniform ( $\vec{V} = \vec{V}_{avg}$ ) across the inlet or outlet, we could simply take it outside the integral. Then we could write the rate of inflow or outflow of momentum through the inlet or outlet in simple algebraic form,

Momentum flow rate across a uniform inlet or outlet:

$$\int_{A_c} \rho \vec{V} (\vec{V} \cdot \vec{n}) dA_c = \rho V_{avg} A_c \vec{V}_{avg} = \dot{m} \vec{V}_{avg} \quad (6-20)$$

The uniform flow approximation is reasonable at some inlets and outlets, e.g., the well-rounded entrance to a pipe, the flow at the entrance to a wind tunnel test section, and a slice through a water jet moving at nearly uniform speed through air (Fig. 6-14). At each such inlet or outlet, Eq. 6-20 can be applied directly.

## Momentum-Flux Correction Factor, $\beta$

Unfortunately, the velocity across most inlets and outlets of practical engineering interest is *not* uniform. Nevertheless, it turns out that we can still convert the control surface integral of Eq. 6-17 into algebraic form, but a dimensionless correction factor  $\beta$ , called the **momentum-flux correction factor**, is required, as first shown by the French scientist Joseph Boussinesq

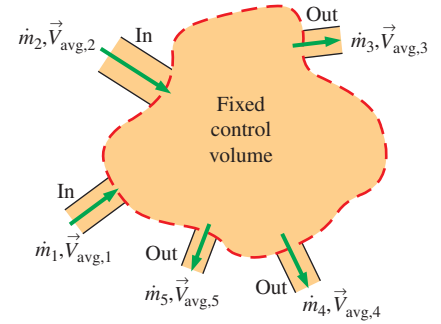
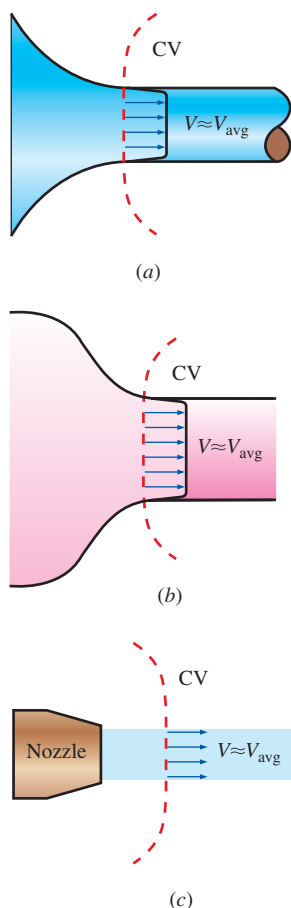


FIGURE 6-13

In a typical engineering problem, the control volume may contain multiple inlets and outlets; at each inlet or outlet we define the mass flow rate  $\dot{m}$  and the average velocity  $\vec{V}_{avg}$ .

**FIGURE 6-14**

Examples of inlets or outlets in which the uniform flow approximation is reasonable: (a) the well-rounded entrance to a pipe, (b) the entrance to a wind tunnel test section, and (c) a slice through a free water jet in air.

(1842–1929). The algebraic form of Eq. 6-17 for a fixed control volume is then written as

$$\sum \vec{F} = \frac{d}{dt} \int_{CV} \rho \vec{V} dV + \sum_{out} \beta \dot{m} \vec{V}_{avg} - \sum_{in} \beta \dot{m} \vec{V}_{avg} \quad (6-21)$$

where a unique value of momentum-flux correction factor is applied to each inlet and outlet in the control surface. Note that  $\beta = 1$  for the case of uniform flow over an inlet or outlet, as in Fig. 6-14. For the general case, we define  $\beta$  such that the integral form of the momentum flux into or out of the control surface at an inlet or outlet of cross-sectional area  $A_c$  can be expressed in terms of mass flow rate  $\dot{m}$  through the inlet or outlet and average velocity  $\vec{V}_{avg}$  through the inlet or outlet,

$$\text{Momentum flux across an inlet or outlet:} \quad \int_{A_c} \rho \vec{V} (\vec{V} \cdot \vec{n}) dA_c = \beta \dot{m} \vec{V}_{avg} \quad (6-22)$$

For the case in which density is uniform over the inlet or outlet and  $\vec{V}$  is in the same direction as  $\vec{V}_{avg}$  over the inlet or outlet, we solve Eq. 6-22 for  $\beta$ ,

$$\beta = \frac{\int_{A_c} \rho V (\vec{V} \cdot \vec{n}) dA_c}{\dot{m} V_{avg}} = \frac{\int_{A_c} \rho V (\vec{V} \cdot \vec{n}) dA_c}{\rho V_{avg} A_c V_{avg}} \quad (6-23)$$

where we have substituted  $\rho V_{avg} A_c$  for  $\dot{m}$  in the denominator. The densities cancel and since  $V_{avg}$  is constant, it can be brought inside the integral. Furthermore, if the control surface slices normal to the inlet or outlet area,  $(\vec{V} \cdot \vec{n}) dA_c = V dA_c$ . Then, Eq. 6-23 simplifies to

$$\text{Momentum-flux correction factor:} \quad \beta = \frac{1}{A_c} \int_{A_c} \left( \frac{V}{V_{avg}} \right)^2 dA_c \quad (6-24)$$

It may be shown that  $\beta$  is always greater than or equal to unity.

### EXAMPLE 6-1 Momentum-Flux Correction Factor for Laminar Pipe Flow

Consider laminar flow through a very long straight section of round pipe. It is shown in Chap. 8 that the velocity profile through a cross-sectional area of the pipe is parabolic (Fig. 6-15), with the axial velocity component given by

$$V = 2V_{avg} \left( 1 - \frac{r^2}{R^2} \right) \quad (1)$$

where  $R$  is the radius of the inner wall of the pipe and  $V_{avg}$  is the average velocity. Calculate the momentum-flux correction factor through a cross section of the pipe for the case in which the pipe flow represents an outlet of the control volume, as sketched in Fig. 6-15.

**SOLUTION** For a given velocity distribution we are to calculate the momentum-flux correction factor.

**Assumptions** 1 The flow is incompressible and steady. 2 The control volume slices through the pipe normal to the pipe axis, as sketched in Fig. 6–15.

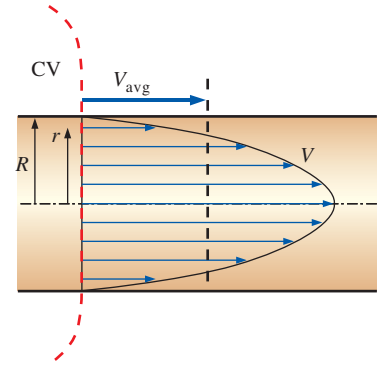
**Analysis** We substitute the given velocity profile for  $V$  in Eq. 6–24 and integrate, noting that  $dA_c = 2\pi r dr$ ,

$$\beta = \frac{1}{A_c} \int_{A_c} \left( \frac{V}{V_{\text{avg}}} \right)^2 dA_c = \frac{4}{\pi R^2} \int_0^R \left( 1 - \frac{r^2}{R^2} \right)^2 2\pi r dr \quad (2)$$

Defining a new integration variable  $y = 1 - r^2/R^2$  and thus  $dy = -2r dr/R^2$  (also,  $y = 1$  at  $r = 0$ , and  $y = 0$  at  $r = R$ ) and performing the integration, the momentum-flux correction factor for fully developed laminar flow becomes

**Laminar flow:** 
$$\beta = -4 \int_1^0 y^2 dy = -4 \left[ \frac{y^3}{3} \right]_1^0 = \frac{4}{3} \quad (3)$$

**Discussion** We have calculated  $\beta$  for an outlet, but the same result would have been obtained if we had considered the cross section of the pipe as an inlet to the control volume.



**FIGURE 6–15**

Velocity profile over a cross section of a pipe in which the flow is fully developed and laminar.

From Example 6–1 we see that  $\beta$  is not very close to unity for fully developed laminar pipe flow, and ignoring  $\beta$  could potentially lead to significant error. If we were to perform the same kind of integration as in Example 6–1 but for fully developed *turbulent* rather than laminar pipe flow, we would find that  $\beta$  ranges from about 1.01 to 1.04. Since these values are so close to unity, many practicing engineers completely disregard the momentum-flux correction factor. While the neglect of  $\beta$  in turbulent flow calculations may have an insignificant effect on the final results, it is wise to keep it in our equations. Doing so not only improves the accuracy of our calculations, but reminds us to include the momentum-flux correction factor when solving laminar flow control volume problems.

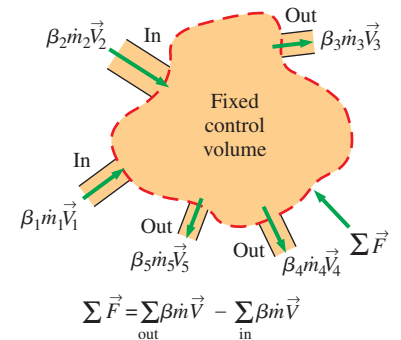
For turbulent flow  $\beta$  may have an insignificant effect at inlets and outlets, but for laminar flow  $\beta$  may be important and should not be neglected. It is wise to include  $\beta$  in all momentum control volume problems.

## Steady Flow

If the flow is also *steady*, the time derivative term in Eq. 6–21 vanishes and we are left with

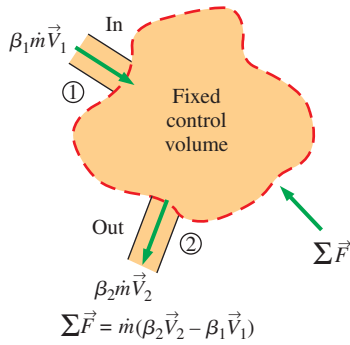
**Steady linear momentum equation:** 
$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V} \quad (6-25)$$

where we have dropped the subscript “avg” from average velocity. Equation 6–25 states that *the net force acting on the control volume during steady flow is equal to the difference between the rates of outgoing and incoming momentum flows*. This statement is illustrated in Fig. 6–16. It can also be expressed for any direction, since Eq. 6–25 is a vector equation.

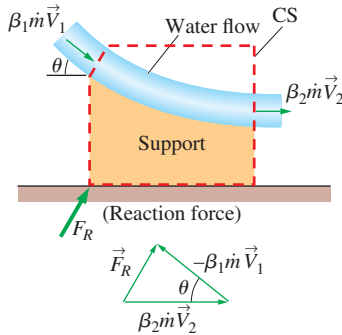


**FIGURE 6–16**

The net force acting on the control volume during steady flow is equal to the difference between the outgoing and the incoming momentum fluxes.

**FIGURE 6-17**

A control volume with only one inlet and one outlet.



Note:  $\vec{V}_2 \neq \vec{V}_1$  even if  $|\vec{V}_2| = |\vec{V}_1|$

**FIGURE 6-18**

The determination by vector addition of the reaction force on the support caused by a change of direction of water.

## Steady Flow with One Inlet and One Outlet

Many practical engineering problems involve just one inlet and one outlet (Fig. 6–17). The mass flow rate for such **single-stream systems** remains constant, and Eq. 6–25 reduces to

$$\text{One inlet and one outlet:} \quad \Sigma \vec{F} = \dot{m}(\beta_2 \vec{V}_2 - \beta_1 \vec{V}_1) \quad (6-26)$$

where we have adopted the usual convention that subscript 1 implies the inlet and subscript 2 the outlet, and  $\vec{V}_1$  and  $\vec{V}_2$  denote the *average* velocities across the inlet and outlet, respectively.

We emphasize again that all the preceding relations are *vector* equations, and thus all the additions and subtractions are *vector* additions and subtractions. Recall that subtracting a vector is equivalent to adding it after reversing its direction (Fig. 6–18). When writing the momentum equation for a specified coordinate direction (such as the  $x$ -axis), we use the projections of the vectors on that axis. For example, Eq. 6–26 is written along the  $x$ -coordinate as

$$\text{Along } x\text{-coordinate:} \quad \Sigma F_x = \dot{m}(\beta_2 V_{2,x} - \beta_1 V_{1,x}) \quad (6-27)$$

where  $\Sigma F_x$  is the vector sum of the  $x$ -components of the forces, and  $V_{2,x}$  and  $V_{1,x}$  are the  $x$ -components of the outlet and inlet velocities of the fluid stream, respectively. The force or velocity components in the positive  $x$ -direction are positive quantities, and those in the negative  $x$ -direction are negative quantities. Also, it is good practice to take the direction of unknown forces in the positive directions (unless the problem is very straightforward). A negative value obtained for an unknown force indicates that the assumed direction is wrong and should be reversed.

## Flow with No External Forces

An interesting situation arises when there are no external forces (such as weight, pressure, and reaction forces) acting on the body in the direction of motion—a common situation for space vehicles and satellites. For a control volume with multiple inlets and outlets, Eq. 6–21 reduces in this case to

$$\text{No external forces:} \quad 0 = \frac{d(m\vec{V})_{CV}}{dt} + \sum_{out} \beta \dot{m} \vec{V} - \sum_{in} \beta \dot{m} \vec{V} \quad (6-28)$$

This is an expression of the conservation of momentum principle, which is stated in words as *in the absence of external forces, the rate of change of the momentum of a control volume is equal to the difference between the rates of incoming and outgoing momentum flow rates*.

When the mass  $m$  of the control volume remains nearly constant, the first term of Eq. 6–28 becomes simply mass times acceleration, since

$$\frac{d(m\vec{V})_{CV}}{dt} = m_{CV} \frac{d\vec{V}_{CV}}{dt} = (m\vec{a})_{CV} = m_{CV} \vec{a}$$

Therefore, the control volume in this case can be treated as a solid body (a fixed-mass system) with a net thrusting force (or just **thrust**) of

$$\text{Thrust:} \quad \vec{F}_{thrust} = m_{body} \vec{a} = \sum_{in} \beta \dot{m} \vec{V} - \sum_{out} \beta \dot{m} \vec{V} \quad (6-29)$$

acting on the body. In Eq. 6–29, fluid velocities are relative to an inertial reference frame—that is, a coordinate system that is fixed in space or is

moving uniformly at constant velocity on a straight path. When analyzing the motion of bodies moving at constant velocity on a straight path, it is convenient to choose an inertial reference frame that moves with the body at the same velocity on the same path. In this case the velocities of fluid streams relative to the inertial reference frame are identical to the velocities relative to the moving body, which are much easier to apply. This approach, while not strictly valid for noninertial reference frames, can also be used to calculate the *initial* acceleration of a space vehicle when its rocket is fired (Fig. 6–19).

Recall that thrust is a mechanical force typically generated through the reaction of an accelerating fluid. In the jet engine of an aircraft, for example, hot exhaust gases are accelerated by the action of expansion and outflow of gases through the back of the engine, and a thrusting force is produced by a reaction in the opposite direction. The generation of thrust is based on Newton's third law of motion, which states that *for every action at a point there is an equal and opposite reaction*. In the case of a jet engine, if the engine exerts a force on exhaust gases, then the exhaust gases exert an equal force on the engine in the opposite direction. That is, the pushing force exerted on the departing gases by the engine is equal to the thrusting force the departing gases exert on the remaining mass of the aircraft in the opposite direction  $\vec{F}_{\text{thrust}} = -\vec{F}_{\text{push}}$ . On the free-body diagram of an aircraft, the effect of outgoing exhaust gases is accounted for by the insertion of a force in the opposite direction of motion of the exhaust gases.

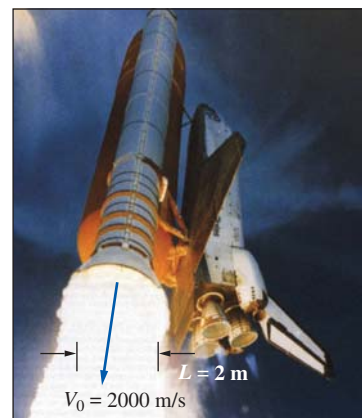


FIGURE 6–19

The thrust needed to lift the space shuttle is generated by the rocket engines as a result of momentum change of the fuel as it is accelerated from about zero to an exit speed of about 2000 m/s after combustion.

NASA

### EXAMPLE 6–2 The Force to Hold a Deflector Elbow in Place

A reducing elbow is used to deflect water flow at a rate of 14 kg/s in a horizontal pipe upward 30° while accelerating it (Fig. 6–20). The elbow discharges water into the atmosphere. The cross-sectional area of the elbow is 113 cm<sup>2</sup> at the inlet and 7 cm<sup>2</sup> at the outlet. The elevation difference between the centers of the outlet and the inlet is 30 cm. The weight of the elbow and the water in it is considered to be negligible. Determine (a) the gage pressure at the center of the inlet of the elbow and (b) the anchoring force needed to hold the elbow in place.

**SOLUTION** A reducing elbow deflects water upward and discharges it to the atmosphere. The pressure at the inlet of the elbow and the force needed to hold the elbow in place are to be determined.

**Assumptions** 1 The flow is steady, and the frictional effects are negligible. 2 The weight of the elbow and the water in it is negligible. 3 The water is discharged to the atmosphere, and thus the gage pressure at the outlet is zero. 4 The flow is turbulent and fully developed at both the inlet and outlet of the control volume, and we take the momentum-flux correction factor to be  $\beta = 1.03$  (as a conservative estimate) at both the inlet and the outlet.

**Properties** We take the density of water to be 1000 kg/m<sup>3</sup>.

**Analysis** (a) We take the elbow as the control volume and designate the inlet by 1 and the outlet by 2. We also take the  $x$ - and  $z$ -coordinates as shown. The continuity equation for this one-inlet, one-outlet, steady-flow system is  $\dot{m}_1 = \dot{m}_2 = \dot{m} = 14$  kg/s. Noting that  $\dot{m} = \rho AV$ , the inlet and outlet velocities of water are

$$V_1 = \frac{\dot{m}}{\rho A_1} = \frac{14 \text{ kg/s}}{(1000 \text{ kg/m}^3)(0.0113 \text{ m}^2)} = 1.24 \text{ m/s}$$

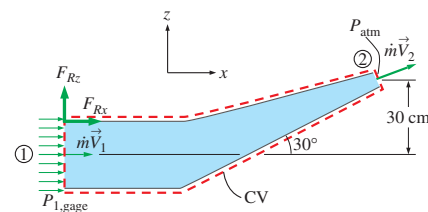


FIGURE 6–20

Schematic for Example 6–2.



$$V_2 = \frac{\dot{m}}{\rho A_2} = \frac{14 \text{ kg/s}}{(1000 \text{ kg/m}^3)(7 \times 10^{-4} \text{ m}^2)} = 20.0 \text{ m/s}$$

We use the Bernoulli equation (Chap. 5) as a first approximation to calculate the pressure. In Chap. 8 we will learn how to account for frictional losses along the walls. Taking the center of the inlet cross section as the reference level ( $z_1 = 0$ ) and noting that  $P_2 = P_{\text{atm}}$ , the Bernoulli equation for a streamline going through the center of the elbow is expressed as

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$P_1 - P_2 = \rho g \left( \frac{V_2^2 - V_1^2}{2g} + z_2 - z_1 \right)$$

$$P_1 - P_{\text{atm}} = (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)$$

$$\times \left( \frac{(20 \text{ m/s})^2 - (1.24 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} + 0.3 - 0 \right) \left( \frac{1 \text{ kN}}{1000 \text{ kg}\cdot\text{m/s}^2} \right)$$

$$P_{1, \text{gage}} = 202.2 \text{ kN/m}^2 = \mathbf{202.2 \text{ kPa}} \quad (\text{gage})$$

(b) The momentum equation for steady flow is

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

We let the  $x$ - and  $z$ -components of the anchoring force of the elbow be  $F_{Rx}$  and  $F_{Rz}$ , and assume them to be in the positive direction. We also use gage pressure since the atmospheric pressure acts on the entire control surface. Then the momentum equations along the  $x$ - and  $z$ -axes become

$$F_{Rx} + P_{1, \text{gage}} A_1 = \beta \dot{m} V_2 \cos \theta - \beta \dot{m} V_1$$

$$F_{Rz} = \beta \dot{m} V_2 \sin \theta$$

where we have set  $\beta = \beta_1 = \beta_2$ . Solving for  $F_{Rx}$  and  $F_{Rz}$ , and substituting the given values,

$$\begin{aligned} F_{Rx} &= \beta \dot{m} (V_2 \cos \theta - V_1) - P_{1, \text{gage}} A_1 \\ &= 1.03(14 \text{ kg/s})[(20 \cos 30^\circ - 1.24) \text{ m/s}] \left( \frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2} \right) \\ &\quad - (202,200 \text{ N/m}^2)(0.0113 \text{ m}^2) \\ &= 232 - 2285 = \mathbf{-2053 \text{ N}} \end{aligned}$$

$$F_{Rz} = \beta \dot{m} V_2 \sin \theta = (1.03)(14 \text{ kg/s})(20 \sin 30^\circ \text{ m/s}) \left( \frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2} \right) = \mathbf{144 \text{ N}}$$

The negative result for  $F_{Rx}$  indicates that the assumed direction is wrong, and it should be reversed. Therefore,  $F_{Rx}$  acts in the negative  $x$ -direction.

**Discussion** There is a nonzero pressure distribution along the inside walls of the elbow, but since the control volume is outside the elbow, these pressures do not appear in our analysis. The weight of the elbow and the water in it could be added to the vertical force for better accuracy. The actual value of  $P_{1, \text{gage}}$  will be higher than that calculated here because of frictional and other irreversible losses in the elbow.

**EXAMPLE 6–3 The Force to Hold a Reversing Elbow in Place**

The deflector elbow in Example 6–2 is replaced by a reversing elbow such that the fluid makes a 180° U-turn before it is discharged, as shown in Fig. 6–21. The elevation difference between the centers of the inlet and the exit sections is still 0.3 m. Determine the anchoring force needed to hold the elbow in place.

**SOLUTION** The inlet and the outlet velocities and the pressure at the inlet of the elbow remain the same, but the vertical component of the anchoring force at the connection of the elbow to the pipe is zero in this case ( $F_{Rz} = 0$ ) since there is no other force or momentum flux in the vertical direction (we are neglecting the weight of the elbow and the water). The horizontal component of the anchoring force is determined from the momentum equation written in the  $x$ -direction. Noting that the outlet velocity is negative since it is in the negative  $x$ -direction, we have

$$F_{Rx} + P_{1,\text{gage}}A_1 = \beta_2\dot{m}(-V_2) - \beta_1\dot{m}V_1 = -\beta\dot{m}(V_2 + V_1)$$

Solving for  $F_{Rx}$  and substituting the known values,

$$\begin{aligned} F_{Rx} &= -\beta\dot{m}(V_2 + V_1) - P_{1,\text{gage}}A_1 \\ &= -(1.03)(14 \text{ kg/s})[(20 + 1.24) \text{ m/s}]\left(\frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2}\right) - (202,200 \text{ N/m}^2)(0.0113 \text{ m}^2) \\ &= -306 - 2285 = \mathbf{-2591 \text{ N}} \end{aligned}$$

Therefore, the horizontal force on the flange is 2591 N acting in the negative  $x$ -direction (the elbow is trying to separate from the pipe). This force is equivalent to the weight of about 260 kg mass, and thus the connectors (such as bolts) used must be strong enough to withstand this force.

**Discussion** The reaction force in the  $x$ -direction is larger than that of Example 6–2 since the walls turn the water over a much greater angle. If the reversing elbow is replaced by a straight nozzle (like one used by firefighters) such that water is discharged in the positive  $x$ -direction, the momentum equation in the  $x$ -direction becomes

$$F_{Rx} + P_{1,\text{gage}}A_1 = \beta\dot{m}V_2 - \beta\dot{m}V_1 \rightarrow F_{Rx} = \beta\dot{m}(V_2 - V_1) - P_{1,\text{gage}}A_1$$

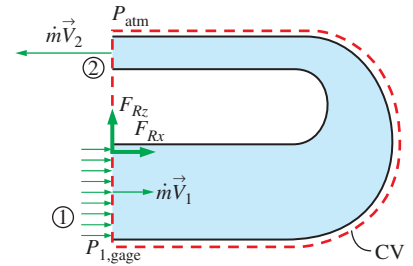
since both  $V_1$  and  $V_2$  are in the positive  $x$ -direction. This shows the importance of using the correct sign (positive if in the positive direction and negative if in the opposite direction) for velocities and forces.

**EXAMPLE 6–4 Water Jet Striking a Stationary Plate**

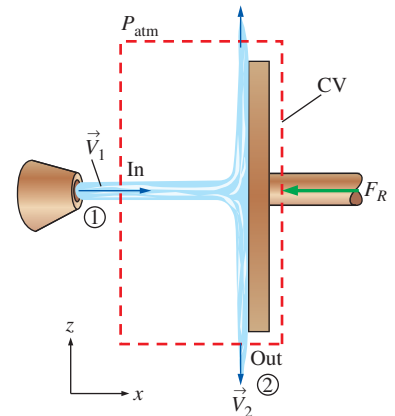
Water is accelerated by a nozzle to an average speed of 20 m/s, and strikes a stationary vertical plate at a rate of 10 kg/s with a normal velocity of 20 m/s (Fig. 6–22). After the strike, the water stream splatters off in all directions in the plane of the plate. Determine the force needed to prevent the plate from moving horizontally due to the water stream.

**SOLUTION** A water jet strikes a vertical stationary plate normally. The force needed to hold the plate in place is to be determined.

**Assumptions** 1 The flow of water at the nozzle outlet is steady. 2 The water splatters in directions normal to the approach direction of the water jet.

**FIGURE 6–21**

Schematic for Example 6–3.

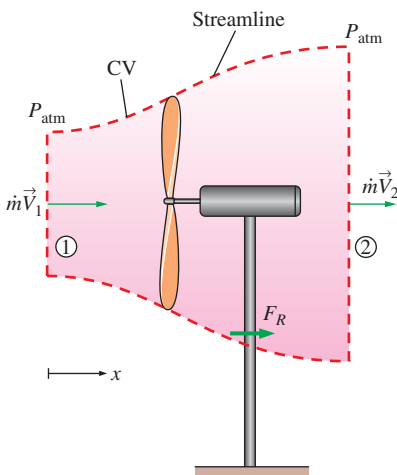
**FIGURE 6–22**

Schematic for Example 6–4.

**FIGURE 6-23**

The downwash of a helicopter is similar to the jet discussed in Example 6-4. The jet impinges on the surface of the water in this case, causing circular waves as seen here.

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**FIGURE 6-24**

Schematic for Example 6-5.

**3** The water jet is exposed to the atmosphere, and thus the pressure of the water jet and the splattered water leaving the control volume is atmospheric pressure, which is disregarded since it acts on the entire system. **4** The vertical forces and momentum fluxes are not considered since they have no effect on the horizontal reaction force. **5** The effect of the momentum-flux correction factor is negligible, and thus  $\beta \cong 1$  at the inlet.

**Analysis** We draw the control volume for this problem such that it contains the entire plate and cuts through the water jet and the support bar normally. The momentum equation for steady flow is given as

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V} \quad (1)$$

Writing Eq. 1 for this problem along the  $x$ -direction (without forgetting the negative sign for forces and velocities in the negative  $x$ -direction) and noting that  $V_{1,x} = V_1$  and  $V_{2,x} = 0$  gives

$$-F_R = 0 - \beta \dot{m} V_1$$

Substituting the given values,

$$F_R = \beta \dot{m} V_1 = (1)(10 \text{ kg/s})(20 \text{ m/s}) \left( \frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2} \right) = \mathbf{200 \text{ N}}$$

Therefore, the support must apply a 200-N horizontal force (equivalent to the weight of about a 20-kg mass) in the negative  $x$ -direction (the opposite direction of the water jet) to hold the plate in place. A similar situation occurs in the downwash of a helicopter (Fig. 6-23).

**Discussion** The plate absorbs the full brunt of the momentum of the water jet since the  $x$ -direction momentum at the outlet of the control volume is zero. If the control volume were drawn instead along the interface between the water and the plate, there would be additional (unknown) pressure forces in the analysis. By cutting the control volume through the support, we avoid having to deal with this additional complexity. This is an example of a “wise” choice of control volume.

### EXAMPLE 6-5 Power Generation and Wind Loading of a Wind Turbine

A wind generator with a 30-ft-diameter blade span has a cut-in wind speed (minimum speed for power generation) of 7 mph, at which velocity the turbine generates 0.4 kW of electric power (Fig. 6-24). Determine (a) the efficiency of the wind turbine-generator unit and (b) the horizontal force exerted by the wind on the supporting mast of the wind turbine. What is the effect of doubling the wind velocity to 14 mph on power generation and the force exerted? Assume the efficiency remains the same, and take the density of air to be  $0.076 \text{ lbm/ft}^3$ .

**SOLUTION** The power generation and loading of a wind turbine are to be analyzed. The efficiency and the force exerted on the mast are to be determined, and the effects of doubling the wind velocity are to be investigated.

**Assumptions** **1** The wind flow is steady and incompressible. **2** The efficiency of the turbine-generator is independent of wind speed. **3** The frictional effects are negligible, and thus none of the incoming kinetic energy is converted to

thermal energy. **4** The average velocity of air through the wind turbine is the same as the wind velocity (actually, it is considerably less—see Chap. 14).

**5** The wind flow is nearly uniform upstream and downstream of the wind turbine and thus the momentum-flux correction factor is  $\beta = \beta_1 = \beta_2 \cong 1$ .

**Properties** The density of air is given to be  $0.076 \text{ lbm/ft}^3$ .

**Analysis** Kinetic energy is a mechanical form of energy, and thus it can be converted to work entirely. Therefore, the power potential of the wind is proportional to its kinetic energy, which is  $V^2/2$  per unit mass, and thus the maximum power is  $\dot{m}V^2/2$  for a given mass flow rate:

$$\begin{aligned} V_1 &= (7 \text{ mph}) \left( \frac{1.4667 \text{ ft/s}}{1 \text{ mph}} \right) = 10.27 \text{ ft/s} \\ \dot{m} &= \rho_1 V_1 A_1 = \rho_1 V_1 \frac{\pi D^2}{4} = (0.076 \text{ lbm/ft}^3)(10.27 \text{ ft/s}) \frac{\pi (30 \text{ ft})^2}{4} = 551.7 \text{ lbm/s} \\ \dot{W}_{\max} &= \dot{m} k e_1 = \dot{m} \frac{V_1^2}{2} \\ &= (551.7 \text{ lbm/s}) \frac{(10.27 \text{ ft/s})^2}{2} \left( \frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2} \right) \left( \frac{1 \text{ kW}}{737.56 \text{ lbf} \cdot \text{ft/s}} \right) \\ &= 1.225 \text{ kW} \end{aligned}$$

Therefore, the available power to the wind turbine is 1.225 kW at the wind velocity of 7 mph. Then the turbine-generator efficiency becomes

$$\eta_{\text{wind turbine}} = \frac{\dot{W}_{\text{act}}}{\dot{W}_{\max}} = \frac{0.4 \text{ kW}}{1.225 \text{ kW}} = \mathbf{0.327} \quad (\text{or } \mathbf{32.7\%})$$

(b) The frictional effects are assumed to be negligible, and thus the portion of incoming kinetic energy not converted to electric power leaves the wind turbine as outgoing kinetic energy. Noting that the mass flow rate remains constant, the exit velocity is determined to be

$$\dot{m} k e_2 = \dot{m} k e_1 (1 - \eta_{\text{wind turbine}}) \rightarrow \dot{m} \frac{V_2^2}{2} = \dot{m} \frac{V_1^2}{2} (1 - \eta_{\text{wind turbine}}) \quad (1)$$

or

$$V_2 = V_1 \sqrt{1 - \eta_{\text{wind turbine}}} = (10.27 \text{ ft/s}) \sqrt{1 - 0.327} = 8.43 \text{ ft/s}$$

To determine the force on the mast (Fig. 6–25), we draw a control volume around the wind turbine such that the wind is normal to the control surface at the inlet and the outlet and the entire control surface is at atmospheric pressure (Fig. 6–23). The momentum equation for steady flow is given as

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V} \quad (2)$$

Writing Eq. 2 along the  $x$ -direction and noting that  $\beta = 1$ ,  $V_{1,x} = V_1$ , and  $V_{2,x} = V_2$  give

$$F_R = \dot{m} V_2 - \dot{m} V_1 = \dot{m} (V_2 - V_1) \quad (3)$$

Substituting the known values into Eq. 3 gives

$$\begin{aligned} F_R &= \dot{m} (V_2 - V_1) = (551.7 \text{ lbm/s})(8.43 - 10.27 \text{ ft/s}) \left( \frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2} \right) \\ &= -31.5 \text{ lbf} \end{aligned}$$



**FIGURE 6–25**

Forces and moments on the supporting mast of a modern wind turbine can be substantial, and increase like  $V^2$ ; thus the mast is typically quite large and strong.

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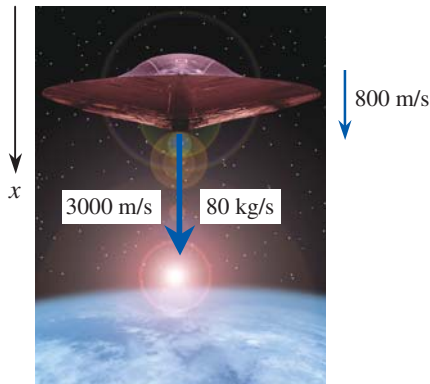


FIGURE 6-26

Schematic for Example 6-6.

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The negative sign indicates that the reaction force acts in the negative  $x$ -direction, as expected. Then the force exerted by the wind on the mast becomes  $F_{\text{mast}} = -F_R = 31.5 \text{ lbf}$ .

The power generated is proportional to  $V^3$  since the mass flow rate is proportional to  $V$  and the kinetic energy to  $V^2$ . Therefore, doubling the wind velocity to 14 mph will increase the power generation by a factor of  $2^3 = 8$  to  $0.4 \times 8 = 3.2 \text{ kW}$ . The force exerted by the wind on the support mast is proportional to  $V^2$ . Therefore, doubling the wind velocity to 14 mph will increase the wind force by a factor of  $2^2 = 4$  to  $31.5 \times 4 = 126 \text{ lbf}$ .

**Discussion** Wind turbines are treated in more detail in Chap. 14.

### EXAMPLE 6-6 Deceleration of a Spacecraft

A spacecraft with a mass of 12,000 kg is dropping vertically towards a planet at a constant speed of 800 m/s (Fig. 6-26). To slow down the spacecraft, a solid-fuel rocket at the bottom is fired, and combustion gases leave the rocket at a constant rate of 80 kg/s and at a velocity of 3000 m/s relative to the spacecraft in the direction of motion of the spacecraft for a period of 5 s. Disregarding the small change in the mass of the spacecraft, determine (a) the deceleration of the spacecraft during this period, (b) the change of velocity of the spacecraft, and (c) the thrust exerted on the spacecraft.

**SOLUTION** The rocket of a spacecraft is fired in the direction of motion. The deceleration, the velocity change, and the thrust are to be determined.

**Assumptions** **1** The flow of combustion gases is steady and one-dimensional during the firing period, but the flight of the spacecraft is unsteady. **2** There are no external forces acting on the spacecraft, and the effect of pressure force at the nozzle outlet is negligible. **3** The mass of discharged fuel is negligible relative to the mass of the spacecraft, and thus, the spacecraft may be treated as a solid body with a constant mass. **4** The nozzle is well designed such that the effect of the momentum-flux correction factor is negligible, and thus,  $\beta \cong 1$ .

**Analysis** (a) For convenience, we choose an inertial reference frame that moves with the spacecraft at the same initial velocity. Then the velocities of the fluid stream relative to an inertial reference frame become simply the velocities relative to the spacecraft. We take the direction of motion of the spacecraft as the positive direction along the  $x$ -axis. There are no external forces acting on the spacecraft, and its mass is essentially constant. Therefore, the spacecraft can be treated as a solid body with constant mass, and the momentum equation in this case is, from Eq. 6-29,

$$\vec{F}_{\text{thrust}} = m_{\text{spacecraft}} \vec{a}_{\text{spacecraft}} = \sum_{\text{in}} \beta \dot{m} \vec{V} - \sum_{\text{out}} \beta \dot{m} \vec{V}$$

where the fluid stream velocities relative to the inertial reference frame in this case are identical to the velocities relative to the spacecraft. Noting that the motion is on a straight line and the discharged gases move in the positive  $x$ -direction, we write the momentum equation using magnitudes as

$$m_{\text{spacecraft}} a_{\text{spacecraft}} = m_{\text{spacecraft}} \frac{dV_{\text{spacecraft}}}{dt} = -\dot{m}_{\text{gas}} V_{\text{gas}}$$



Noting that gases leave in the positive  $x$ -direction and substituting, the acceleration of the spacecraft during the first 5 seconds is determined to be

$$a_{\text{spacecraft}} = \frac{dV_{\text{spacecraft}}}{dt} = -\frac{\dot{m}_{\text{gas}}}{m_{\text{spacecraft}}} V_{\text{gas}} = -\frac{80 \text{ kg/s}}{12,000 \text{ kg}}(+3000 \text{ m/s}) = -20 \text{ m/s}^2$$

The negative value confirms that the spacecraft is decelerating in the positive  $x$  direction at a rate of  $20 \text{ m/s}^2$ .

(b) Knowing the deceleration, which is constant, the velocity change of the spacecraft during the first 5 seconds is determined from the definition of acceleration to be

$$\begin{aligned} dV_{\text{spacecraft}} &= a_{\text{spacecraft}} dt \rightarrow \Delta V_{\text{spacecraft}} = a_{\text{spacecraft}} \Delta t = (-20 \text{ m/s}^2)(5 \text{ s}) \\ &= -100 \text{ m/s} \end{aligned}$$

(c) The thrusting force exerted on the space aircraft is, from Eq. 6-29,

$$F_{\text{thrust}} = 0 - \dot{m}_{\text{gas}} V_{\text{gas}} = 0 - (80 \text{ kg/s})(+3000 \text{ m/s}) \left( \frac{1 \text{ kN}}{1000 \text{ kg}\cdot\text{m/s}^2} \right) = -240 \text{ kN}$$

The negative sign indicates that the thrusting force due to firing of the rocket acts on the aircraft in the negative  $x$ -direction.

**Discussion** Note that if this fired rocket were attached somewhere on a test stand, it would exert a force of 240 kN (equivalent to the weight of about 24 tons of mass) to its support in the opposite direction of the discharged gases.

### EXAMPLE 6-7 Net Force on a Flange

Water flows at a rate of 18.5 gal/min through a flanged faucet with a partially closed gate valve spigot (Fig. 6-27). The inner diameter of the pipe at the location of the flange is 0.780 in ( $= 0.0650 \text{ ft}$ ), and the pressure at that location is measured to be 13.0 psig. The total weight of the faucet assembly plus the water within it is 12.8 lbf. Calculate the net force on the flange.

**SOLUTION** Water flow through a flanged faucet is considered. The net force acting on the flange is to be calculated.

**Assumptions** **1** The flow is steady and incompressible. **2** The flow at the inlet and at the outlet is turbulent and fully developed so that the momentum-flux correction factor is about 1.03. **3** The pipe diameter at the outlet of the faucet is the same as that at the flange.

**Properties** The density of water at room temperature is  $62.3 \text{ lbm/ft}^3$ .

**Analysis** We choose the faucet and its immediate surroundings as the control volume, as shown in Fig. 6-27 along with all the forces acting on it. These forces include the weight of the water and the weight of the faucet assembly, the gage pressure force at the inlet to the control volume, and the net force of the flange on the control volume, which we call  $\vec{F}_R$ . We use gage pressure for convenience since the gage pressure on the rest of the control surface is zero (atmospheric pressure). Note that the pressure through the outlet of the control volume is also atmospheric since we are assuming incompressible flow; hence, the gage pressure is also zero through the outlet.

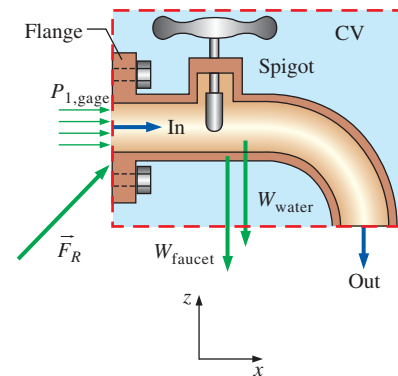


FIGURE 6-27

Control volume for Example 6-7 with all forces shown; gage pressure is used for convenience.

We now apply the control volume conservation laws. Conservation of mass is trivial here since there is only one inlet and one outlet; namely, the mass flow rate into the control volume is equal to the mass flow rate out of the control volume. Also, the outflow and inflow average velocities are identical since the inner diameter is constant and the water is incompressible, and are determined to be

$$V_2 = V_1 = V = \frac{\dot{V}}{A_c} = \frac{\dot{V}}{\pi D^2/4} = \frac{18.5 \text{ gal/min}}{\pi(0.065 \text{ ft})^2/4} \left( \frac{0.1337 \text{ ft}^3}{1 \text{ gal}} \right) \left( \frac{1 \text{ min}}{60 \text{ s}} \right) = 12.42 \text{ ft/s}$$

Also,

$$\dot{m} = \rho \dot{V} = (62.3 \text{ lbm/ft}^3)(18.5 \text{ gal/min}) \left( \frac{0.1337 \text{ ft}^3}{1 \text{ gal}} \right) \left( \frac{1 \text{ min}}{60 \text{ s}} \right) = 2.568 \text{ lbm/s}$$

Next we apply the momentum equation for steady flow,

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V} \quad (1)$$

We let the  $x$ - and  $z$ -components of the force acting on the flange be  $F_{Rx}$  and  $F_{Rz}$ , and assume them to be in the positive directions. The magnitude of the velocity in the  $x$ -direction is  $+V_1$  at the inlet, but zero at the outlet. The magnitude of the velocity in the  $z$ -direction is zero at the inlet, but  $-V_2$  at the outlet. Also, the weight of the faucet assembly and the water within it acts in the  $-z$ -direction as a body force. No pressure or viscous forces act on the chosen (wise) control volume in the  $z$ -direction.

The components of Eq. 1 along the  $x$ - and  $z$ -directions become

$$F_{Rx} + P_{1, \text{gage}} A_1 = 0 - \dot{m}(+V_1)$$

$$F_{Rz} - W_{\text{faucet}} - W_{\text{water}} = \dot{m}(-V_2) - 0$$

Solving for  $F_{Rx}$  and  $F_{Rz}$ , and substituting the given values,

$$\begin{aligned} F_{Rx} &= -\dot{m}V_1 - P_{1, \text{gage}} A_1 \\ &= -(2.568 \text{ lbm/s})(12.42 \text{ ft/s}) \left( \frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2} \right) - (13 \text{ lbf/in}^2) \frac{\pi(0.780 \text{ in})^2}{4} \\ &= -7.20 \text{ lbf} \end{aligned}$$

$$\begin{aligned} F_{Rz} &= -\dot{m}V_2 + W_{\text{faucet} + \text{water}} \\ &= -(2.568 \text{ lbm/s})(12.42 \text{ ft/s}) \left( \frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2} \right) + 12.8 \text{ lbf} = 11.8 \text{ lbf} \end{aligned}$$

Then the net force of the flange on the control volume is expressed in vector form as

$$\vec{F}_R = F_{Rx} \vec{i} + F_{Rz} \vec{k} = -7.20 \vec{i} + 11.8 \vec{k} \text{ lbf}$$

From Newton's third law, the force the faucet assembly exerts on the flange is the negative of  $\vec{F}_R$ ,

$$\vec{F}_{\text{faucet on flange}} = -\vec{F}_R = \mathbf{7.20 \vec{i} - 11.8 \vec{k} \text{ lbf}}$$

**Discussion** The faucet assembly pulls to the right and down; this agrees with our intuition. Namely, the water exerts a high pressure at the inlet, but

the outlet pressure is atmospheric. In addition, the momentum of the water at the inlet in the  $x$ -direction is lost in the turn, causing an additional force to the right on the pipe walls. The faucet assembly weighs much more than the momentum effect of the water, so we expect the force to be downward. Note that labeling forces such as “faucet on flange” clarifies the direction of the force.

## 6-5 ■ REVIEW OF ROTATIONAL MOTION AND ANGULAR MOMENTUM

The motion of a rigid body can be considered to be the combination of translational motion of its center of mass and rotational motion about its center of mass. The translational motion is analyzed using the linear momentum equation, Eq. 6-1. Now we discuss the rotational motion—a motion during which all points in the body move in circles about the axis of rotation. Rotational motion is described with angular quantities such as angular distance  $\theta$ , angular velocity  $\vec{\omega}$ , and angular acceleration  $\vec{\alpha}$ .

The amount of rotation of a point in a body is expressed in terms of the angle  $\theta$  swept by a line of length  $r$  that connects the point to the axis of rotation and is perpendicular to the axis. The angle  $\theta$  is expressed in radians (rad), which is the arc length corresponding to  $\theta$  on a circle of unit radius. Noting that the circumference of a circle of radius  $r$  is  $2\pi r$ , the angular distance traveled by any point in a rigid body during a complete rotation is  $2\pi$  rad. The physical distance traveled by a point along its circular path is  $l = \theta r$ , where  $r$  is the normal distance of the point from the axis of rotation and  $\theta$  is the angular distance in rad. Note that 1 rad corresponds to  $360/(2\pi) \approx 57.3^\circ$ .

The magnitude of angular velocity  $\omega$  is the angular distance traveled per unit time, and the magnitude of angular acceleration  $\alpha$  is the rate of change of angular velocity. They are expressed as (Fig. 6-28),

$$\omega = \frac{d\theta}{dt} = \frac{d(l/r)}{dt} = \frac{1}{r} \frac{dl}{dt} = \frac{V}{r} \quad \text{and} \quad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{1}{r} \frac{dV}{dt} = \frac{a_t}{r} \quad (6-30)$$

or

$$V = r\omega \quad \text{and} \quad a_t = r\alpha \quad (6-31)$$

where  $V$  is the linear velocity and  $a_t$  is the linear acceleration in the tangential direction for a point located at a distance  $r$  from the axis of rotation. Note that  $\omega$  and  $\alpha$  are the same for all points of a rotating rigid body, but  $V$  and  $a_t$  are not (they are proportional to  $r$ ).

Newton’s second law requires that there must be a force acting in the tangential direction to cause angular acceleration. The strength of the rotating effect, called the *moment* or *torque*, is proportional to the magnitude of the force and its distance from the axis of rotation. The perpendicular distance from the axis of rotation to the line of action of the force is called the *moment arm*, and the magnitude of torque  $M$  acting on a point mass  $m$  at normal distance  $r$  from the axis of rotation is expressed as

$$M = rF_t = rma_t = mr^2\alpha \quad (6-32)$$

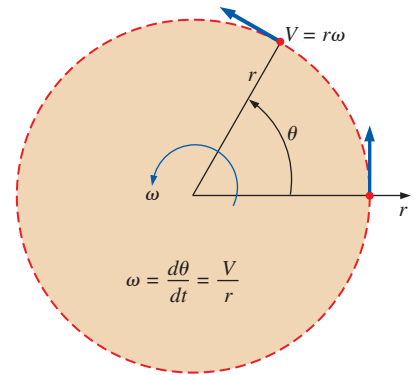


FIGURE 6-28

The relations between angular distance  $\theta$ , angular velocity  $\omega$ , and linear velocity  $V$  in a plane.

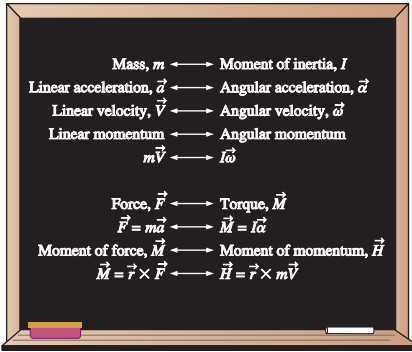


FIGURE 6-29

Analogy between corresponding linear and angular quantities.

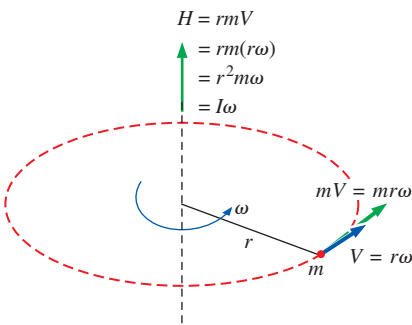


FIGURE 6-30

Angular momentum of point mass  $m$  rotating at angular velocity  $\omega$  at distance  $r$  from the axis of rotation.

The total torque acting on a rotating rigid body about an axis is determined by integrating the torque acting on differential mass  $\delta m$  over the entire body to give

$$\text{Magnitude of torque: } M = \int_{\text{mass}} r^2 \alpha \, \delta m = \left[ \int_{\text{mass}} r^2 \, \delta m \right] \alpha = I \alpha \quad (6-33)$$

where  $I$  is the *moment of inertia* of the body about the axis of rotation, which is a measure of the inertia of a body against rotation. The relation  $M = I \alpha$  is the counterpart of Newton's second law, with torque replacing force, moment of inertia replacing mass, and angular acceleration replacing linear acceleration (Fig. 6-29). Note that unlike mass, the rotational inertia of a body also depends on the distribution of the mass of the body with respect to the axis of rotation. Therefore, a body whose mass is closely packed about its axis of rotation has a small resistance against angular acceleration, while a body whose mass is concentrated at its periphery has a large resistance against angular acceleration. A flywheel is a good example of the latter.

The linear momentum of a body of mass  $m$  having velocity  $\vec{V}$  is  $m\vec{V}$ , and the direction of linear momentum is identical to the direction of velocity. Noting that the moment of a force is equal to the product of the force and the normal distance, the magnitude of the moment of momentum, called the **angular momentum**, of a point mass  $m$  about an axis is expressed as  $H = rmV = r^2 m \omega$ , where  $r$  is the normal distance from the axis of rotation to the line of action of the momentum vector (Fig. 6-30). Then the total angular momentum of a rotating rigid body is determined by integration to be

$$\text{Magnitude of angular momentum: } H = \int_{\text{mass}} r^2 \omega \, \delta m = \left[ \int_{\text{mass}} r^2 \, \delta m \right] \omega = I \omega \quad (6-34)$$

where again  $I$  is the *moment of inertia* of the body about the axis of rotation. It can also be expressed more generally in vector form as

$$\vec{H} = I \vec{\omega} \quad (6-35)$$

Note that the angular velocity  $\vec{\omega}$  is the same at every point of a rigid body.

Newton's second law  $\vec{F} = m\vec{a}$  was expressed in terms of the rate of change of linear momentum in Eq. 6-1 as  $\vec{F} = d(m\vec{V})/dt$ . Likewise, the counterpart of Newton's second law for rotating bodies  $\vec{M} = I\vec{\alpha}$  is expressed in Eq. 6-2 in terms of the rate of change of angular momentum as

$$\text{Angular momentum equation: } \vec{M} = I\vec{\alpha} = I \frac{d\vec{\omega}}{dt} = \frac{d(I\vec{\omega})}{dt} = \frac{d\vec{H}}{dt} \quad (6-36)$$

where  $\vec{M}$  is the net torque applied on the body about the axis of rotation.

The angular velocity of rotating machinery is typically expressed in rpm (number of revolutions per minute) and denoted by  $\dot{n}$ . Noting that velocity is distance traveled per unit time and the angular distance traveled during each revolution is  $2\pi$ , the angular velocity of rotating machinery is  $\omega = 2\pi\dot{n}$  rad/min or

$$\text{Angular velocity versus rpm: } \omega = 2\pi\dot{n} \text{ (rad/min)} = \frac{2\pi\dot{n}}{60} \text{ (rad/s)} \quad (6-37)$$

Consider a constant force  $F$  acting in the tangential direction on the outer surface of a shaft of radius  $r$  rotating at an rpm of  $\dot{n}$ . Noting that work  $W$  is

force times distance, and power  $\dot{W}$  is work done per unit time and thus force times velocity, we have  $\dot{W}_{\text{shaft}} = FV = Fr\omega = M\omega$ . Therefore, the power transmitted by a shaft rotating at an rpm of  $\dot{n}$  under the influence of an applied torque  $M$  is (Fig. 6–31)

**Shaft power:** 
$$\dot{W}_{\text{shaft}} = \omega M = 2\pi\dot{n}M \quad (6-38)$$

The kinetic energy of a body of mass  $m$  during translational motion is  $\text{KE} = \frac{1}{2}mV^2$ . Noting that  $V = r\omega$ , the rotational kinetic energy of a body of mass  $m$  at a distance  $r$  from the axis of rotation is  $\text{KE} = \frac{1}{2}mr^2\omega^2$ . The total rotational kinetic energy of a rotating rigid body about an axis is determined by integrating the rotational kinetic energies of differential masses  $dm$  over the entire body to give

**Rotational kinetic energy:** 
$$\text{KE}_r = \frac{1}{2}I\omega^2 \quad (6-39)$$

where again  $I$  is the moment of inertia of the body and  $\omega$  is the angular velocity.

During rotational motion, the direction of velocity changes even when its magnitude remains constant. Velocity is a vector quantity, and thus a change in direction constitutes a change in velocity with time, and thus acceleration. This is called **centripetal acceleration**. Its magnitude is

$$a_r = \frac{V^2}{r} = r\omega^2$$

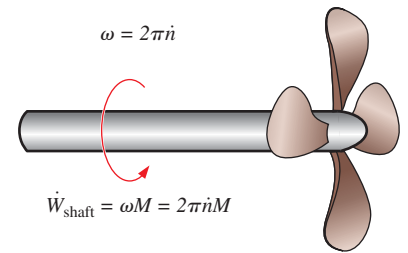
Centripetal acceleration is directed toward the axis of rotation (opposite direction of radial acceleration), and thus the radial acceleration is negative. Noting that acceleration is a constant multiple of force, centripetal acceleration is the result of a force acting on the body toward the axis of rotation, known as the **centripetal force**, whose magnitude is  $F_r = mV^2/r$ . Tangential and radial accelerations are perpendicular to each other (since the radial and tangential directions are perpendicular), and the total linear acceleration is determined by their vector sum,  $\vec{a} = \vec{a}_t + \vec{a}_r$ . For a body rotating at constant angular velocity, the only acceleration is the centripetal acceleration. The centripetal force does not produce torque since its line of action intersects the axis of rotation.

## 6–6 ■ THE ANGULAR MOMENTUM EQUATION

The linear momentum equation discussed in Section 6–4 is useful for determining the relationship between the linear momentum of flow streams and the resultant forces. Many engineering problems involve the moment of the linear momentum of flow streams, and the rotational effects caused by them. Such problems are best analyzed by the *angular momentum equation*, also called the *moment of momentum equation*. An important class of fluid devices, called *turbomachines*, which include centrifugal pumps, turbines, and fans, is analyzed by the angular momentum equation.

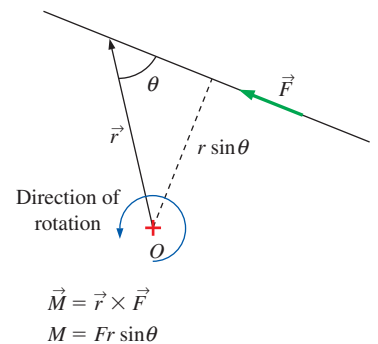
The *moment of a force*  $\vec{F}$  about a point  $O$  is the vector (or cross) product (Fig. 6–32)

**Moment of a force:** 
$$\vec{M} = \vec{r} \times \vec{F} \quad (6-40)$$



**FIGURE 6–31**

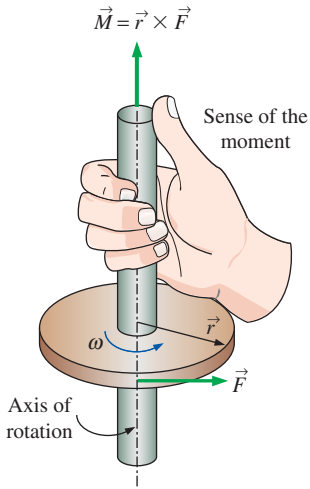
The relations between angular velocity, rpm, and the power transmitted through a rotating shaft.



**FIGURE 6–32**

The moment of a force  $\vec{F}$  about a point  $O$  is the vector product of the position vector  $\vec{r}$  and  $\vec{F}$ .



**FIGURE 6-33**

The determination of the direction of the moment by the right-hand rule.

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt} \int_{\text{CV}} \rho b \, dV + \int_{\text{CS}} \rho b (\vec{V}_r \cdot \vec{n}) \, dA$$

$$B = \vec{H} \quad b = \vec{r} \times \vec{V} \quad b = \vec{r} \times \vec{V}$$

$$\frac{d\vec{H}_{\text{sys}}}{dt} = \frac{d}{dt} \int_{\text{CV}} (\vec{r} \times \vec{V}) \rho \, dV + \int_{\text{CS}} (\vec{r} \times \vec{V}) \rho (\vec{V}_r \cdot \vec{n}) \, dA$$

**FIGURE 6-34**

The angular momentum equation is obtained by replacing  $B$  in the Reynolds transport theorem by the angular momentum  $\vec{H}$ , and  $b$  by the angular momentum per unit mass  $\vec{r} \times \vec{V}$ .

where  $\vec{r}$  is the position vector from point  $O$  to any point on the line of action of  $\vec{F}$ . The vector product of two vectors is a vector whose line of action is normal to the plane that contains the crossed vectors ( $\vec{r}$  and  $\vec{F}$  in this case) and whose magnitude is

$$\text{Magnitude of the moment of a force:} \quad M = Fr \sin \theta \quad (6-41)$$

where  $\theta$  is the angle between the lines of action of the vectors  $\vec{r}$  and  $\vec{F}$ . Therefore, the magnitude of the moment about point  $O$  is equal to the magnitude of the force multiplied by the normal distance of the line of action of the force from the point  $O$ . The sense of the moment vector  $\vec{M}$  is determined by the right-hand rule: when the fingers of the right hand are curled in the direction that the force tends to cause rotation, the thumb points the direction of the moment vector (Fig. 6-33). Note that a force whose line of action passes through point  $O$  produces zero moment about point  $O$ .

The vector product of  $\vec{r}$  and the momentum vector  $m\vec{V}$  gives the *moment of momentum*, also called the *angular momentum*, about a point  $O$  as

$$\text{Moment of momentum:} \quad \vec{H} = \vec{r} \times m\vec{V} \quad (6-42)$$

Therefore,  $\vec{r} \times \vec{V}$  represents the angular momentum per unit mass, and the angular momentum of a differential mass  $\delta m = \rho \, dV$  is  $d\vec{H} = (\vec{r} \times \vec{V}) \rho \, dV$ . Then the angular momentum of a system is determined by integration to be

$$\text{Moment of momentum (system):} \quad \vec{H}_{\text{sys}} = \int_{\text{sys}} (\vec{r} \times \vec{V}) \rho \, dV \quad (6-43)$$

The rate of change of the moment of momentum is

$$\text{Rate of change of moment of momentum:} \quad \frac{d\vec{H}_{\text{sys}}}{dt} = \frac{d}{dt} \int_{\text{sys}} (\vec{r} \times \vec{V}) \rho \, dV \quad (6-44)$$

The angular momentum equation for a system was expressed in Eq. 6-2 as

$$\sum \vec{M} = \frac{d\vec{H}_{\text{sys}}}{dt} \quad (6-45)$$

where  $\sum \vec{M} = \sum (\vec{r} \times \vec{F})$  is the net torque or moment applied on the system, which is the vector sum of the moments of all forces acting on the system, and  $d\vec{H}_{\text{sys}}/dt$  is the rate of change of the angular momentum of the system. Equation 6-45 is stated as *the rate of change of angular momentum of a system is equal to the net torque acting on the system*. This equation is valid for a fixed quantity of mass and an inertial reference frame, i.e., a reference frame that is fixed or moves with a constant velocity in a straight path.

The general control volume formulation of the angular momentum equation is obtained by setting  $b = \vec{r} \times \vec{V}$  and thus  $B = \vec{H}$  in the general Reynolds transport theorem. It gives (Fig. 6-34)

$$\frac{d\vec{H}_{\text{sys}}}{dt} = \frac{d}{dt} \int_{\text{CV}} (\vec{r} \times \vec{V}) \rho \, dV + \int_{\text{CS}} (\vec{r} \times \vec{V}) \rho (\vec{V}_r \cdot \vec{n}) \, dA \quad (6-46)$$

The left-hand side of this equation is, from Eq. 6-45, equal to  $\sum \vec{M}$ . Substituting, the angular momentum equation for a general control volume (stationary or moving, fixed shape or distorting) is

$$\text{General:} \quad \sum \vec{M} = \frac{d}{dt} \int_{\text{CV}} (\vec{r} \times \vec{V}) \rho \, dV + \int_{\text{CS}} (\vec{r} \times \vec{V}) \rho (\vec{V}_r \cdot \vec{n}) \, dA \quad (6-47)$$

which is stated in words as

$$\left( \begin{array}{l} \text{The sum of all} \\ \text{external moments} \\ \text{acting on a CV} \end{array} \right) = \left( \begin{array}{l} \text{The time rate of change} \\ \text{of the angular momentum} \\ \text{of the contents of the CV} \end{array} \right) + \left( \begin{array}{l} \text{The net flow rate of} \\ \text{angular momentum} \\ \text{out of the control} \\ \text{surface by mass flow} \end{array} \right)$$

Again,  $\vec{V}_r = \vec{V} - \vec{V}_{CS}$  is the fluid velocity relative to the control surface (for use in mass flow rate calculations at all locations where the fluid crosses the control surface), and  $\vec{V}$  is the fluid velocity as viewed from a fixed reference frame. The product  $\rho(\vec{V}_r \cdot \vec{n}) dA$  represents the mass flow rate through  $dA$  into or out of the control volume, depending on the sign.

For a fixed control volume (no motion or deformation of the control volume),  $\vec{V}_r = \vec{V}$  and the angular momentum equation becomes

$$\text{Fixed CV:} \quad \sum \vec{M} = \frac{d}{dt} \int_{CV} (\vec{r} \times \vec{V}) \rho dV + \int_{CS} (\vec{r} \times \vec{V}) \rho (\vec{V} \cdot \vec{n}) dA \quad (6-48)$$

Also, note that the forces acting on the control volume consist of *body forces* that act throughout the entire body of the control volume such as gravity, and *surface forces* that act on the control surface such as the pressure and reaction forces at points of contact. The net torque consists of the moments of these forces as well as the torques applied on the control volume.

## Special Cases

During *steady flow*, the amount of angular momentum within the control volume remains constant, and thus the time rate of change of angular momentum of the contents of the control volume is zero. Then,

$$\text{Steady flow:} \quad \sum \vec{M} = \int_{CS} (\vec{r} \times \vec{V}) \rho (\vec{V} \cdot \vec{n}) dA \quad (6-49)$$

In many practical applications, the fluid crosses the boundaries of the control volume at a certain number of inlets and outlets, and it is convenient to replace the area integral by an algebraic expression written in terms of the average properties over the cross-sectional areas where the fluid enters or leaves the control volume. In such cases, the angular momentum flow rate can be expressed as the difference in the angular momentum of outgoing and incoming streams. Furthermore, in many cases the moment arm  $\vec{r}$  is either constant along the inlet or outlet (as in radial flow turbomachines) or is large compared to the diameter of the inlet or outlet pipe (as in rotating lawn sprinklers, Fig. 6–35). In such cases, the *average* value of  $\vec{r}$  is used throughout the cross-sectional area of the inlet or outlet. Then, an approximate form of the angular momentum equation in terms of average properties at inlets and outlets becomes

$$\sum \vec{M} \cong \frac{d}{dt} \int_{CV} (\vec{r} \times \vec{V}) \rho dV + \sum_{out} (\vec{r} \times \dot{m} \vec{V}) - \sum_{in} (\vec{r} \times \dot{m} \vec{V}) \quad (6-50)$$

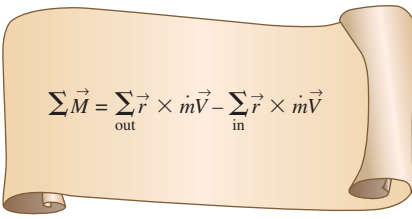
You may be wondering why we don't introduce a correction factor into Eq. 6–50, like we did for conservation of energy (Chap. 5) and for conservation of linear momentum (Section 6–4). The reason is that the cross product of  $\vec{r}$  and  $\dot{m} \vec{V}$  is dependent on problem geometry, and thus, such a correction



FIGURE 6–35

A rotating lawn sprinkler is a good example of application of the angular momentum equation.

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**FIGURE 6-36**

The net torque acting on a control volume during steady flow is equal to the difference between the outgoing and incoming angular momentum flow rates.

factor would vary from problem to problem. Therefore, whereas we can readily calculate a kinetic energy flux correction factor and a momentum flux correction factor for fully developed pipe flow that can be applied to various problems, we cannot do so for angular momentum. Fortunately, in many problems of practical engineering interest, the error associated with using average values of radius and velocity is small, and the approximation of Eq. 6-50 is reasonable.

If the flow is *steady*, Eq. 6-50 further reduces to (Fig. 6-36)

$$\text{Steady flow:} \quad \sum \vec{M} = \sum_{\text{out}} (\vec{r} \times \dot{m} \vec{V}) - \sum_{\text{in}} (\vec{r} \times \dot{m} \vec{V}) \quad (6-51)$$

Equation 6-51 states that *the net torque acting on the control volume during steady flow is equal to the difference between the outgoing and incoming angular momentum flow rates*. This statement can also be expressed for any specified direction. Note that velocity  $\vec{V}$  in Eq. 6-51 is the velocity relative to an inertial coordinate system.

In many problems, all the significant forces and momentum flows are in the same plane, and thus all give rise to moments in the same plane and about the same axis. For such cases, Eq. 6-51 can be expressed in scalar form as

$$\sum M = \sum_{\text{out}} r \dot{m} V - \sum_{\text{in}} r \dot{m} V \quad (6-52)$$

where  $r$  represents the average normal distance between the point about which moments are taken and the line of action of the force or velocity, provided that the sign convention for the moments is observed. That is, all moments in the counterclockwise direction are positive, and all moments in the clockwise direction are negative.

## Flow with No External Moments

When there are no external moments applied, the angular momentum equation Eq. 6-50 reduces to

$$\text{No external moments:} \quad 0 = \frac{d\vec{H}_{\text{cv}}}{dt} + \sum_{\text{out}} (\vec{r} \times \dot{m} \vec{V}) - \sum_{\text{in}} (\vec{r} \times \dot{m} \vec{V}) \quad (6-53)$$

This is an expression of the conservation of angular momentum principle, which can be stated as *in the absence of external moments, the rate of change of the angular momentum of a control volume is equal to the difference between the incoming and outgoing angular momentum fluxes*.

When the moment of inertia  $I$  of the control volume remains constant, the first term on the right side of Eq. 6-53 becomes simply moment of inertia times angular acceleration,  $I\vec{\alpha}$ . Therefore, the control volume in this case can be treated as a solid body, with a net torque of

$$\vec{M}_{\text{body}} = I_{\text{body}} \vec{\alpha} = \sum_{\text{in}} (\vec{r} \times \dot{m} \vec{V}) - \sum_{\text{out}} (\vec{r} \times \dot{m} \vec{V}) \quad (6-54)$$

(due to a change of angular momentum) acting on it. This approach can be used to determine the angular acceleration of space vehicles and aircraft when a rocket is fired in a direction different than the direction of motion.

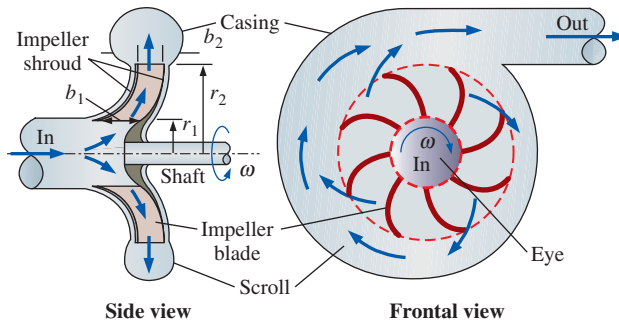


FIGURE 6-37

Side and frontal views of a typical centrifugal pump.

## Radial-Flow Devices

Many rotary-flow devices such as centrifugal pumps and fans involve flow in the radial direction normal to the axis of rotation and are called *radial-flow devices* (Chap. 14). In a centrifugal pump, for example, the fluid enters the device in the axial direction through the eye of the impeller, turns outward as it flows through the passages between the blades of the impeller, collects in the scroll, and is discharged in the tangential direction, as shown in Fig. 6-37. Axial-flow devices are easily analyzed using the linear momentum equation. But radial-flow devices involve large changes in angular momentum of the fluid and are best analyzed with the help of the angular momentum equation.

To analyze a centrifugal pump, we choose the annular region that encloses the impeller section as the control volume, as shown in Fig. 6-38. Note that the average flow velocity, in general, has normal and tangential components at both the inlet and the outlet of the impeller section. Also, when the shaft rotates at angular velocity  $\omega$ , the impeller blades have tangential velocity  $\omega r_1$  at the inlet and  $\omega r_2$  at the outlet. For steady, incompressible flow, the conservation of mass equation is written as

$$\dot{V}_1 = \dot{V}_2 = \dot{V} \quad \rightarrow \quad (2\pi r_1 b_1)V_{1,n} = (2\pi r_2 b_2)V_{2,n} \quad (6-55)$$

where  $b_1$  and  $b_2$  are the flow widths at the inlet where  $r = r_1$  and at the outlet where  $r = r_2$ , respectively. (Note that the actual circumferential cross-sectional area is somewhat less than  $2\pi r b$  since the blade thickness is not zero.) Then the average normal components  $V_{1,n}$  and  $V_{2,n}$  of absolute velocity can be expressed in terms of the volumetric flow rate  $\dot{V}$  as

$$V_{1,n} = \frac{\dot{V}}{2\pi r_1 b_1} \quad \text{and} \quad V_{2,n} = \frac{\dot{V}}{2\pi r_2 b_2} \quad (6-56)$$

The normal velocity components  $V_{1,n}$  and  $V_{2,n}$  as well as pressure acting on the inner and outer circumferential areas pass through the shaft center, and thus they do not contribute to torque about the origin. Then only the tangential velocity components contribute to torque, and the application of the angular momentum equation  $\sum M = \sum_{\text{out}} r \dot{m} V - \sum_{\text{in}} r \dot{m} V$  to the control volume gives

*Euler's turbine equation:*  $T_{\text{shaft}} = \dot{m}(r_2 V_{2,t} - r_1 V_{1,t}) \quad (6-57)$

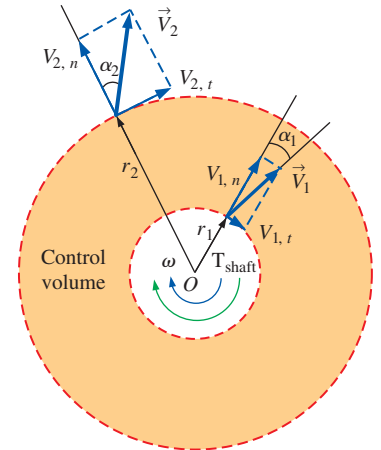


FIGURE 6-38

An annular control volume that encloses the impeller section of a centrifugal pump.

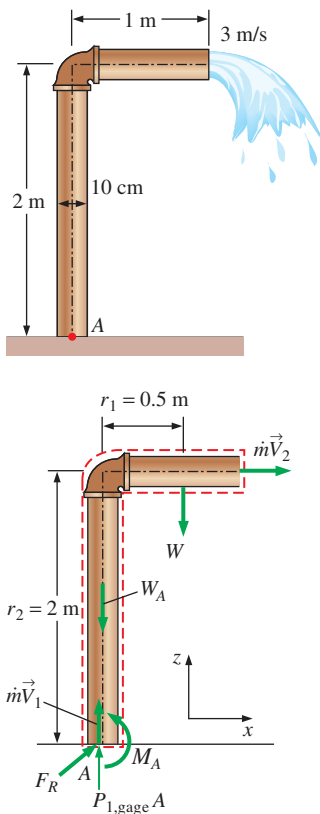
which is known as **Euler's turbine equation**. When the angles  $\alpha_1$  and  $\alpha_2$  between the direction of absolute flow velocities and the radial direction are known, Eq. 6-57 becomes

$$T_{\text{shaft}} = \dot{m}(r_2 V_2 \sin \alpha_2 - r_1 V_1 \sin \alpha_1) \quad (6-58)$$

In the idealized case of the tangential fluid velocity being equal to the blade angular velocity both at the inlet and the exit, we have  $V_{1,t} = \omega r_1$  and  $V_{2,t} = \omega r_2$ , and the torque becomes

$$T_{\text{shaft, ideal}} = \dot{m}\omega(r_2^2 - r_1^2) \quad (6-59)$$

where  $\omega = 2\pi\dot{n}$  is the angular velocity of the blades. When the torque is known, the shaft power is determined from  $\dot{W}_{\text{shaft}} = \omega T_{\text{shaft}} = 2\pi\dot{n}T_{\text{shaft}}$ .



**FIGURE 6-39**

Schematic for Example 6-8 and the free-body diagram.

### EXAMPLE 6-8 Bending Moment Acting at the Base of a Water Pipe

Underground water is pumped through a 10-cm-diameter pipe that consists of a 2-m-long vertical and 1-m-long horizontal section, as shown in Fig. 6-39. Water discharges to atmospheric air at an average velocity of 3 m/s, and the mass of the horizontal pipe section when filled with water is 12 kg per meter length. The pipe is anchored on the ground by a concrete base. Determine the bending moment acting at the base of the pipe (point A) and the required length of the horizontal section that would make the moment at point A zero.

**SOLUTION** Water is pumped through a piping section. The moment acting at the base and the required length of the horizontal section to make this moment zero is to be determined.

**Assumptions** 1 The flow is steady. 2 The water is discharged to the atmosphere, and thus the gage pressure at the outlet is zero. 3 The pipe diameter is small compared to the moment arm, and thus we use average values of radius and velocity at the outlet.

**Properties** We take the density of water to be  $1000 \text{ kg/m}^3$ .

**Analysis** We take the entire L-shaped pipe as the control volume, and designate the inlet by 1 and the outlet by 2. We also take the  $x$ - and  $z$ -coordinates as shown. The control volume and the reference frame are fixed.

The conservation of mass equation for this one-inlet, one-outlet, steady-flow system is  $\dot{m}_1 = \dot{m}_2 = \dot{m}$ , and  $V_1 = V_2 = V$  since  $A_c = \text{constant}$ . The mass flow rate and the weight of the horizontal section of the pipe are

$$\dot{m} = \rho A_c V = (1000 \text{ kg/m}^3)[\pi(0.10 \text{ m})^2/4](3 \text{ m/s}) = 23.56 \text{ kg/s}$$

$$W = mg = (12 \text{ kg/m})(1 \text{ m})(9.81 \text{ m/s}^2)\left(\frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2}\right) = 117.7 \text{ N}$$

To determine the moment acting on the pipe at point A, we need to take the moment of all forces and momentum flows about that point. This is a steady-flow problem, and all forces and momentum flows are in the same plane. Therefore, the angular momentum equation in this case is expressed as

$$\sum M = \sum_{\text{out}} r \dot{m} V - \sum_{\text{in}} r \dot{m} V$$

where  $r$  is the average moment arm,  $V$  is the average speed, all moments in the counterclockwise direction are positive, and all moments in the clockwise direction are negative.



The free-body diagram of the L-shaped pipe is given in Fig. 6–39. Noting that the moments of all forces and momentum flows passing through point  $A$  are zero, the only force that yields a moment about point  $A$  is the weight  $W$  of the horizontal pipe section, and the only momentum flow that yields a moment is the outlet stream (both are negative since both moments are in the clockwise direction). Then the angular momentum equation about point  $A$  becomes

$$M_A - r_1 W = -r_2 \dot{m} V_2$$

Solving for  $M_A$  and substituting give

$$\begin{aligned} M_A &= r_1 W - r_2 \dot{m} V_2 \\ &= (0.5 \text{ m})(118 \text{ N}) - (2 \text{ m})(23.56 \text{ kg/s})(3 \text{ m/s}) \left( \frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2} \right) \\ &= \mathbf{-82.5 \text{ N} \cdot \text{m}} \end{aligned}$$

The negative sign indicates that the assumed direction for  $M_A$  is wrong and should be reversed. Therefore, a moment of 82.5 N·m acts at the stem of the pipe in the clockwise direction. That is, the concrete base must apply a 82.5 N·m moment on the pipe stem in the clockwise direction to counteract the excess moment caused by the exit stream.

The weight of the horizontal pipe is  $w = W/L = 117.7 \text{ N per m length}$ . Therefore, the weight for a length of  $L$  m is  $Lw$  with a moment arm of  $r_1 = L/2$ . Setting  $M_A = 0$  and substituting, the length  $L$  of the horizontal pipe that would cause the moment at the pipe stem to vanish is determined to be

$$0 = r_1 W - r_2 \dot{m} V_2 \rightarrow 0 = (L/2)Lw - r_2 \dot{m} V_2$$

or

$$L = \sqrt{\frac{2r_2 \dot{m} V_2}{w}} = \sqrt{\frac{2(2 \text{ m})(23.56 \text{ kg/s})(3 \text{ m/s})}{117.7 \text{ N/m}}} \left( \frac{\text{N}}{\text{kg} \cdot \text{m/s}^2} \right) = \mathbf{1.55 \text{ m}}$$

**Discussion** Note that the pipe weight and the momentum of the exit stream cause opposing moments at point  $A$ . This example shows the importance of accounting for the moments of momentums of flow streams when performing a dynamic analysis and evaluating the stresses in pipe materials at critical cross sections.

### EXAMPLE 6–9 Power Generation from a Sprinkler System

A large lawn sprinkler (Fig. 6–40) with four identical arms is to be converted into a turbine to generate electric power by attaching a generator to its rotating head, as shown in Fig. 6–41. Water enters the sprinkler from the base along the axis of rotation at a rate of 20 L/s and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 300 rpm in a horizontal plane. The diameter of each jet is 1 cm, and the normal distance between the axis of rotation and the center of each nozzle is 0.6 m. Estimate the electric power produced.

**SOLUTION** A four-armed sprinkler is used to generate electric power. For a specified flow rate and rotational speed, the power produced is to be determined.



**FIGURE 6–40**

Lawn sprinklers often have rotating heads to spread the water over a large area.

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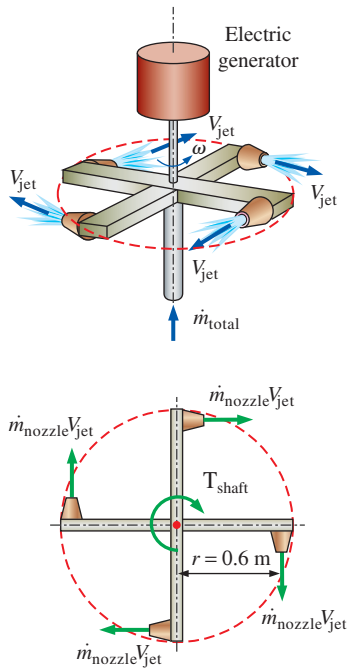


FIGURE 6-41

Schematic for Example 6–9 and the free-body diagram.

**Assumptions** 1 The flow is cyclically steady (i.e., steady from a frame of reference rotating with the sprinkler head). 2 The water is discharged to the atmosphere, and thus the gage pressure at the nozzle exit is zero. 3 Generator losses and air drag of rotating components are neglected. 4 The nozzle diameter is small compared to the moment arm, and thus we use average values of radius and velocity at the outlet.

**Properties** We take the density of water to be  $1000 \text{ kg/m}^3 = 1 \text{ kg/L}$ .

**Analysis** We take the disk that encloses the sprinkler arms as the control volume, which is a stationary control volume.

The conservation of mass equation for this steady-flow system is  $\dot{m}_1 = \dot{m}_2 = \dot{m}_{\text{total}}$ . Noting that the four nozzles are identical, we have  $\dot{m}_{\text{nozzle}} = \dot{m}_{\text{total}}/4$  or  $\dot{V}_{\text{nozzle}} = \dot{V}_{\text{total}}/4$  since the density of water is constant. The average jet exit velocity relative to the rotating nozzle is

$$V_{\text{jet},r} = \frac{\dot{V}_{\text{nozzle}}}{A_{\text{jet}}} = \frac{5 \text{ L/s}}{[\pi(0.01 \text{ m})^2/4]} \left( \frac{1 \text{ m}^3}{1000 \text{ L}} \right) = 63.66 \text{ m/s}$$

The angular and tangential velocities of the nozzles are

$$\omega = 2\pi\dot{n} = 2\pi(300 \text{ rev/min}) \left( \frac{1 \text{ min}}{60 \text{ s}} \right) = 31.42 \text{ rad/s}$$

$$V_{\text{nozzle}} = r\omega = (0.6 \text{ m})(31.42 \text{ rad/s}) = 18.85 \text{ m/s}$$

Note that water in the nozzle is also moving at an average velocity of 18.85 m/s in the opposite direction when it is discharged. The average absolute velocity of the water jet (velocity relative to a fixed location on earth) is the vector sum of its relative velocity (jet velocity relative to the nozzle) and the absolute nozzle velocity,

$$\vec{V}_{\text{jet}} = \vec{V}_{\text{jet},r} + \vec{V}_{\text{nozzle}}$$

All of these three velocities are in the tangential direction, and taking the direction of jet flow as positive, the vector equation can be written in scalar form using magnitudes as

$$V_{\text{jet}} = V_{\text{jet},r} - V_{\text{nozzle}} = 63.66 - 18.85 = 44.81 \text{ m/s}$$

Noting that this is a cyclically steady-flow problem, and all forces and momentum flows are in the same plane, the angular momentum equation is approximated as  $\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$ , where  $r$  is the moment arm, all moments in the counterclockwise direction are positive, and all moments in the clockwise direction are negative.

The free-body diagram of the disk that contains the sprinkler arms is given in Fig. 6–41. Note that the moments of all forces and momentum flows passing through the axis of rotation are zero. The momentum flows via the water jets leaving the nozzles yield a moment in the clockwise direction and the effect of the generator on the control volume is a moment also in the clockwise direction (thus both are negative). Then the angular momentum equation about the axis of rotation becomes

$$-T_{\text{shaft}} = -4r\dot{m}_{\text{nozzle}}V_{\text{jet}} \quad \text{or} \quad T_{\text{shaft}} = r\dot{m}_{\text{total}}V_{\text{jet}}$$

Substituting, the torque transmitted through the shaft is

$$T_{\text{shaft}} = r\dot{m}_{\text{total}}V_{\text{jet}} = (0.6 \text{ m})(20 \text{ kg/s})(44.81 \text{ m/s}) \left( \frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2} \right) = 537.7 \text{ N}\cdot\text{m}$$

since  $\dot{m}_{\text{total}} = \rho \dot{V}_{\text{total}} = (1 \text{ kg/L})(20 \text{ L/s}) = 20 \text{ kg/s}$ .

Then the power generated becomes

$$\dot{W} = \omega T_{\text{shaft}} = (31.42 \text{ rad/s})(537.7 \text{ N}\cdot\text{m}) \left( \frac{1 \text{ kW}}{1000 \text{ N}\cdot\text{m/s}} \right) = \mathbf{16.9 \text{ kW}}$$

Therefore, this sprinkler-type turbine has the potential to produce 16.9 kW of power.

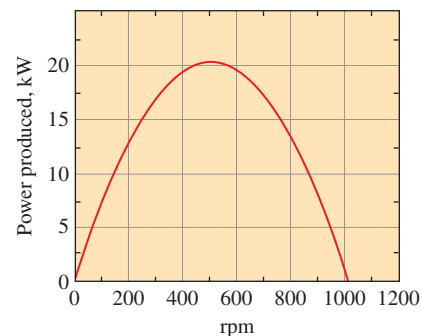
**Discussion** To put the result obtained in perspective, we consider two limiting cases. In the first limiting case, the sprinkler is stuck, and thus, the angular velocity is zero. The torque developed is maximum in this case, since  $V_{\text{nozzle}} = 0$ . Thus  $V_{\text{jet}} = V_{\text{jet}, r} = 63.66 \text{ m/s}$ , giving  $T_{\text{shaft}, \text{max}} = 764 \text{ N}\cdot\text{m}$ . The power generated is zero since the generator shaft does not rotate.

In the second limiting case, the sprinkler shaft is disconnected from the generator (and thus both the useful torque and power generation are zero), and the shaft accelerates until it reaches an equilibrium velocity. Setting  $T_{\text{shaft}} = 0$  in the angular momentum equation gives the absolute water-jet velocity (jet velocity relative to an observer on earth) to be zero,  $V_{\text{jet}} = 0$ . Therefore, the relative velocity  $V_{\text{jet}, r}$  and absolute velocity  $V_{\text{nozzle}}$  are equal but in opposite direction. So, the absolute tangential velocity of the jet (and thus torque) is zero, and the water mass drops straight down like a waterfall under gravity with zero angular momentum (around the axis of rotation). The angular speed of the sprinkler in this case is

$$\dot{n} = \frac{\omega}{2\pi} = \frac{V_{\text{nozzle}}}{2\pi r} = \frac{63.66 \text{ m/s}}{2\pi(0.6 \text{ m})} \left( \frac{60 \text{ s}}{1 \text{ min}} \right) = 1013 \text{ rpm}$$

Of course, the  $T_{\text{shaft}} = 0$  case is possible only for an ideal, frictionless nozzle (i.e., 100 percent nozzle efficiency, as a no-load ideal turbine). Otherwise, there would be a resisting torque due to friction of the water, shaft, and surrounding air.

The variation of power produced with angular speed is plotted in Fig. 6–42. Note that the power produced increases with increasing rpm, reaches a maximum (at about 500 rpm in this case), and then decreases. The actual power produced would be less than this due to generator inefficiency (Chap. 5) and other irreversible losses such as fluid friction within the nozzle (Chap. 8), shaft friction, and aerodynamic drag (Chap. 11).



**FIGURE 6–42**

The variation of power produced with angular speed for the turbine of Example 6–9.



### APPLICATION SPOTLIGHT ■ Manta Ray Swimming

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Aquatic animals propel themselves using a wide variety of mechanisms. Most fish flap their tail to produce thrust, and in doing so they shed two single vortices per flapping cycle, creating a wake that resembles a reverse von Kármán vortex street. The non-dimensional number that describes this vortex shedding is the Strouhal number  $St$ , where  $St = fA/U_\infty$ , where  $f$  is the frequency of actuation,  $A$  is the peak-to-peak amplitude of the trailing edge motion at the half-span, and  $U_\infty$  is the steady swimming velocity. Remarkably, a wide variety of fish and mammals swim in the range  $0.2 < St < 0.35$ .

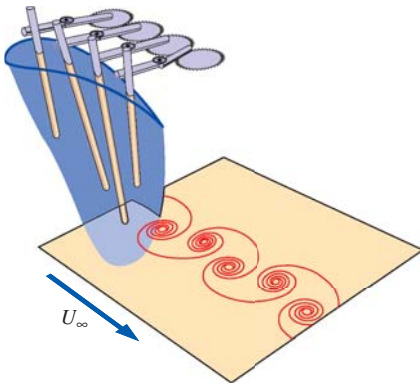
In manta rays (Fig. 6–43), propulsion is achieved by combining oscillatory and undulatory motions of flexible pectoral fins. That is, as the manta ray



**FIGURE 6–43**

The manta ray is the largest of the rays, reaching up to 8 m in span. They swim with a motion that is a combination of flapping and undulation of their large pectoral fins.

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**FIGURE 6-44**

Manta ray fin mechanism, showing the vortex pattern produced in the wake when it is swimming in a range where two single vortices are shed into the wake per flapping cycle. The artificial flexible fin is actuated by four rigid spars; by changing the relative phase differences between adjacent actuators, undulations of varying wavelength can be produced.

**FIGURE 6-45**

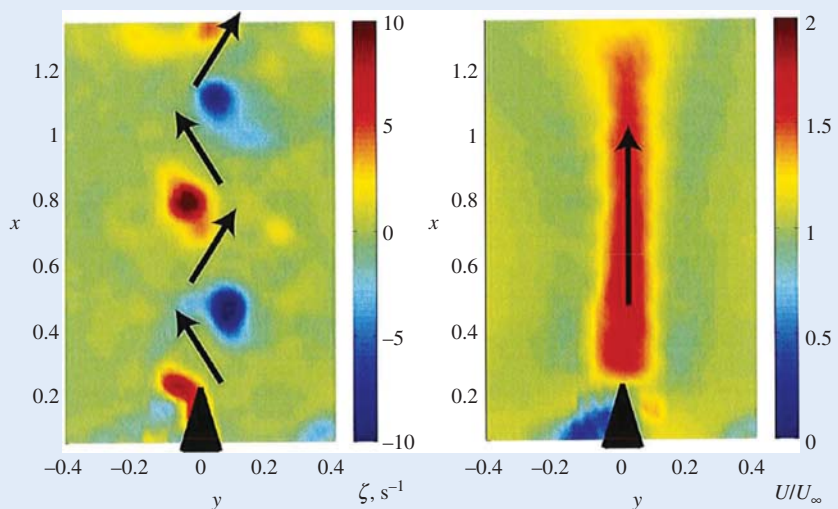
Measurements of the wake of the manta ray fin mechanism, with the flow going from bottom to top. On the left, we see the vortices shed in the wake, alternating between positive vorticity (red) and negative vorticity (blue). The induced velocities are shown by the black arrows, and in this case we see that thrust is being produced. On the right, we see the time-averaged velocity field. The unsteady velocity field induced by the vortices produces a high velocity jet in the time-averaged field. The momentum flux associated with this jet contributes to the total thrust on the fin.

Image courtesy of Peter Dewey, Keith Moored and Alexander Smits. Used by permission.

flaps its fins, it is also generating a traveling wave motion along the chord, opposite to the direction of its motion. This wave motion is not readily apparent because the wavelength is 6 to 10 times greater than the chord length. A similar undulation is observed in sting rays, but there it is more obvious because the wavelength is less than the chord length. Field observations indicate that many species of manta ray are migratory, and that they are very efficient swimmers. They are difficult to study in the laboratory because they are a protected and somewhat fragile creature. However, it is possible to study many aspects of their swimming behavior by mimicking their propulsive techniques using robots or mechanical devices such as that shown in Fig. 6-44. The flow field generated by such a fin displays the vortex shedding seen in other fish studies, and when time-averaged displays a high momentum jet that contributes to the thrust (Fig 6-45). The thrust and efficiencies can also be measured directly, and it appears that the undulatory motion due to the traveling wave is most important to thrust production at high efficiency in the manta ray.

### References

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## SUMMARY

This chapter deals mainly with the conservation of momentum for finite control volumes. The forces acting on the control volume consist of *body forces* that act throughout the entire body of the control volume (such as gravity, electric, and magnetic forces) and *surface forces* that act on the control surface (such as the pressure forces and reaction forces at points of contact). The sum of all forces acting on the control volume at a particular instant in time is represented by  $\Sigma \vec{F}$  and is expressed as

$$\underbrace{\Sigma \vec{F}}_{\text{total force}} = \underbrace{\Sigma \vec{F}_{\text{gravity}}}_{\text{body force}} + \underbrace{\Sigma \vec{F}_{\text{pressure}} + \Sigma \vec{F}_{\text{viscous}} + \Sigma \vec{F}_{\text{other}}}_{\text{surface forces}}$$

Newton's second law can be stated as *the sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system*. Setting  $b = \vec{V}$  and thus  $B = m\vec{V}$  in the Reynolds transport theorem and utilizing Newton's second law gives the *linear momentum equation* for a control volume as

$$\Sigma \vec{F} = \frac{d}{dt} \int_{\text{CV}} \rho \vec{V} dV + \int_{\text{CS}} \rho \vec{V} (\vec{V}_r \cdot \vec{n}) dA$$

which reduces to the following special cases:

*Steady flow:*  $\Sigma \vec{F} = \int_{\text{CS}} \rho \vec{V} (\vec{V}_r \cdot \vec{n}) dA$

*Unsteady flow (algebraic form):*

$$\Sigma \vec{F} = \frac{d}{dt} \int_{\text{CV}} \rho \vec{V} dV + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

*Steady flow (algebraic form):*  $\Sigma \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$

*No external forces:*  $0 = \frac{d(m\vec{V})_{\text{CV}}}{dt} + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$

where  $\beta$  is the momentum-flux correction factor. A control volume whose mass  $m$  remains constant can be treated as a solid body (a fixed-mass system) with a *net thrusting force* (also called simply the *thrust*) of

$$\vec{F}_{\text{thrust}} = m_{\text{CV}} \vec{a} = \sum_{\text{in}} \beta \dot{m} \vec{V} - \sum_{\text{out}} \beta \dot{m} \vec{V}$$

acting on the body.

Newton's second law can also be stated as *the rate of change of angular momentum of a system is equal to the net torque acting on the system*. Setting  $b = \vec{r} \times \vec{V}$  and thus  $B = \vec{H}$  in the general Reynolds transport theorem gives the *angular momentum equation* as

$$\Sigma \vec{M} = \frac{d}{dt} \int_{\text{CV}} (\vec{r} \times \vec{V}) \rho dV + \int_{\text{CS}} (\vec{r} \times \vec{V}) \rho (\vec{V}_r \cdot \vec{n}) dA$$

which reduces to the following special cases:

*Steady flow:*  $\Sigma \vec{M} = \int_{\text{CS}} (\vec{r} \times \vec{V}) \rho (\vec{V}_r \cdot \vec{n}) dA$

*Unsteady flow (algebraic form):*

$$\Sigma \vec{M} = \frac{d}{dt} \int_{\text{CV}} (\vec{r} \times \vec{V}) \rho dV + \sum_{\text{out}} \vec{r} \times \dot{m} \vec{V} - \sum_{\text{in}} \vec{r} \times \dot{m} \vec{V}$$

*Steady and uniform flow:*

$$\Sigma \vec{M} = \sum_{\text{out}} \vec{r} \times \dot{m} \vec{V} - \sum_{\text{in}} \vec{r} \times \dot{m} \vec{V}$$

*Scalar form for one direction:*

$$\Sigma M = \sum_{\text{out}} r \dot{m} V - \sum_{\text{in}} r \dot{m} V$$

*No external moments:*

$$0 = \frac{d\vec{H}_{\text{CV}}}{dt} + \sum_{\text{out}} \vec{r} \times \dot{m} \vec{V} - \sum_{\text{in}} \vec{r} \times \dot{m} \vec{V}$$

A control volume whose moment of inertia  $I$  remains constant can be treated as a solid body (a fixed-mass system), with a net torque of

$$\vec{M}_{\text{CV}} = I_{\text{CV}} \vec{\alpha} = \sum_{\text{in}} \vec{r} \times \dot{m} \vec{V} - \sum_{\text{out}} \vec{r} \times \dot{m} \vec{V}$$

acting on the body. This relation is used to determine the angular acceleration of a spacecraft when a rocket is fired.

The linear and angular momentum equations are of fundamental importance in the analysis of turbomachinery and are used extensively in Chap. 14.

## REFERENCES AND SUGGESTED READING

1. P. K. Kundu, I. M. Cohen, and D. R. Dowling. *Fluid Mechanics*, ed. 5. San Diego, CA: Academic Press, 2011.
2. Terry Wright, *Fluid Machinery: Performance, Analysis, and Design*, Boca Raton, FL: CRC Press, 1999.



## PROBLEMS\*

## Newton's Laws and Conservation of Momentum

**6-1C** Express Newton's first, second, and third laws.

**6-2C** Express Newton's second law of motion for rotating bodies. What can you say about the angular velocity and angular momentum of a rotating nonrigid body of constant mass if the net torque acting on it is zero?

**6-3C** Is momentum a vector? If so, in what direction does it point?

**6-4C** Express the conservation of momentum principle. What can you say about the momentum of a body if the net force acting on it is zero?

## Linear Momentum Equation

**6-5C** Two firefighters are fighting a fire with identical water hoses and nozzles, except that one is holding the hose straight so that the water leaves the nozzle in the same direction it comes, while the other holds it backward so that the water makes a U-turn before being discharged. Which firefighter will experience a greater reaction force?

**6-6C** How do surface forces arise in the momentum analysis of a control volume? How can we minimize the number of surface forces exposed during analysis?



**6-7C** Explain the importance of the Reynolds transport theorem in fluid mechanics, and describe how the linear momentum equation is obtained from it.

**6-8C** What is the importance of the momentum-flux correction factor in the momentum analysis of flow systems? For which type(s) of flow is it significant and must it be considered in analysis: laminar flow, turbulent flow, or jet flow?

**6-9C** Write the momentum equation for steady one-dimensional flow for the case of no external forces and explain the physical significance of its terms.

**6-10C** In the application of the momentum equation, explain why we can usually disregard the atmospheric pressure and work with gage pressures only.

**6-11C** A rocket in space (no friction or resistance to motion) can expel gases relative to itself at some high velocity  $V$ . Is  $V$  the upper limit to the rocket's ultimate velocity?

\* Problems designated by a "C" are concept questions, and students are encouraged to answer them all. Problems designated by an "E" are in English units, and the SI users can ignore them. Problems with the  icon are solved using EES, and complete solutions together with parametric studies are included on the text website. Problems with the  icon are comprehensive in nature and are intended to be solved with an equation solver such as EES.

**6-12C** Describe in terms of momentum and airflow how a helicopter is able to hover.



FIGURE P6-12C

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**6-13C** Does it take more, equal, or less power for a helicopter to hover at the top of a high mountain than it does at sea level? Explain.

**6-14C** In a given location, would a helicopter require more energy in summer or winter to achieve a specified performance? Explain.

**6-15C** A horizontal water jet from a nozzle of constant exit cross section impinges normally on a stationary vertical flat plate. A certain force  $F$  is required to hold the plate against the water stream. If the water velocity is doubled, will the necessary holding force also be doubled? Explain.

**6-16C** Describe body forces and surface forces, and explain how the net force acting on a control volume is determined. Is fluid weight a body force or surface force? How about pressure?

**6-17C** A constant-velocity horizontal water jet from a stationary nozzle impinges normally on a vertical flat plate that rides on a nearly frictionless track. As the water jet hits the plate, it begins to move due to the water force. Will the acceleration of the plate remain constant or change? Explain.

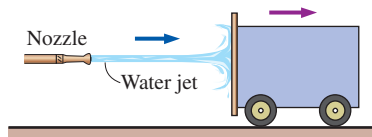


FIGURE P6-17C

**6-18C** A horizontal water jet of constant velocity  $V$  from a stationary nozzle impinges normally on a vertical flat plate that rides on a nearly frictionless track. As the water jet hits



the plate, it begins to move due to the water force. What is the highest velocity the plate can attain? Explain.

**6-19** Water enters a 10-cm-diameter pipe steadily with a uniform velocity of 3 m/s and exits with the turbulent flow velocity distribution given by  $u = u_{\max} (1 - r/R)^{1/7}$ . If the pressure drop along the pipe is 10 kPa, determine the drag force exerted on the pipe by water flow.

**6-20** A 2.5-cm-diameter horizontal water jet with a speed of  $V_j = 40$  m/s relative to the ground is deflected by a  $60^\circ$  stationary cone whose base diameter is 25 cm. Water velocity along the cone varies linearly from zero at the cone surface to the incoming jet speed of 40 m/s at the free surface. Disregarding the effect of gravity and the shear forces, determine the horizontal force  $F$  needed to hold the cone stationary.

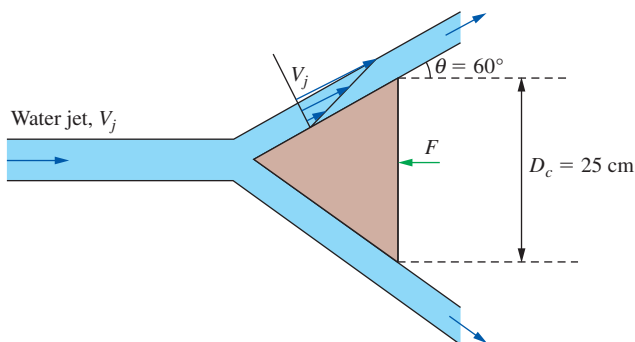


FIGURE P6-20

**6-21** A horizontal water jet of constant velocity  $V$  impinges normally on a vertical flat plate and splashes off the sides in the vertical plane. The plate is moving toward the oncoming water jet with velocity  $\frac{1}{2}V$ . If a force  $F$  is required to maintain the plate stationary, how much force is required to move the plate toward the water jet?

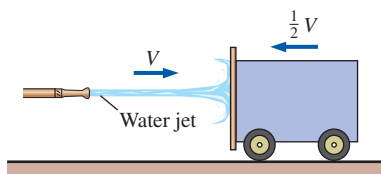


FIGURE P6-21

**6-22** A  $90^\circ$  elbow in a horizontal pipe is used to direct water flow upward at a rate of 40 kg/s. The diameter of the entire elbow is 10 cm. The elbow discharges water into the atmosphere, and thus the pressure at the exit is the local atmospheric pressure. The elevation difference between the centers of the exit and the inlet of the elbow is 50 cm. The weight of the elbow and the water in it is considered to be

negligible. Determine (a) the gage pressure at the center of the inlet of the elbow and (b) the anchoring force needed to hold the elbow in place. Take the momentum-flux correction factor to be 1.03 at both the inlet and the outlet.

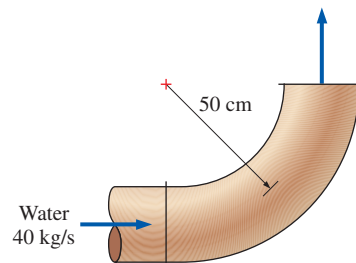


FIGURE P6-22

**6-23** Repeat Prob. 6-22 for the case of another (identical) elbow attached to the existing elbow so that the fluid makes a U-turn. *Answers: (a) 9.81 kPa, (b) 497 N*

**6-24E** A horizontal water jet impinges against a vertical flat plate at 25 ft/s and splashes off the sides in the vertical plane. If a horizontal force of 350 lbf is required to hold the plate against the water stream, determine the volume flow rate of the water.

**6-25** A reducing elbow in a horizontal pipe is used to deflect water flow by an angle  $\theta = 45^\circ$  from the flow direction while accelerating it. The elbow discharges water into the atmosphere. The cross-sectional area of the elbow is  $150 \text{ cm}^2$  at the inlet and  $25 \text{ cm}^2$  at the exit. The elevation difference between the centers of the exit and the inlet is 40 cm. The mass of the elbow and the water in it is 50 kg. Determine the anchoring force needed to hold the elbow in place. Take the momentum-flux correction factor to be 1.03 at both the inlet and outlet.

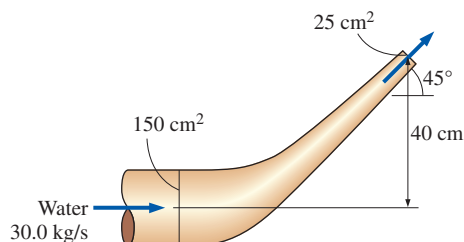


FIGURE P6-25

**6-26** Repeat Prob. 6-25 for the case of  $\theta = 110^\circ$ .

**6-27** Water accelerated by a nozzle to 35 m/s strikes the vertical back surface of a cart moving horizontally at a constant velocity of 10 m/s in the flow direction. The mass flow rate of water through the stationary nozzle is 30 kg/s. After the strike, the water stream splatters off in all directions in

the plane of the back surface. (a) Determine the force that needs to be applied by the brakes of the cart to prevent it from accelerating. (b) If this force were used to generate power instead of wasting it on the brakes, determine the maximum amount of power that could ideally be generated.

Answers: (a)  $-536 \text{ N}$ , (b)  $5.36 \text{ kW}$

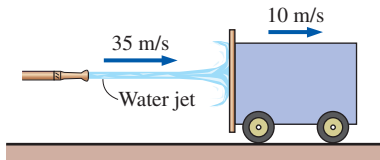


FIGURE P6-27

**6-28** Reconsider Prob. 6-27. If the mass of the cart is  $400 \text{ kg}$  and the brakes fail, determine the acceleration of the cart when the water first strikes it. Assume the mass of water that wets the back surface is negligible.

**6-29E** A  $100\text{-ft}^3/\text{s}$  water jet is moving in the positive  $x$ -direction at  $18 \text{ ft/s}$ . The stream hits a stationary splitter, such that half of the flow is diverted upward at  $45^\circ$  and the other half is directed downward, and both streams have a final average speed of  $18 \text{ ft/s}$ . Disregarding gravitational effects, determine the  $x$ - and  $z$ -components of the force required to hold the splitter in place against the water force.

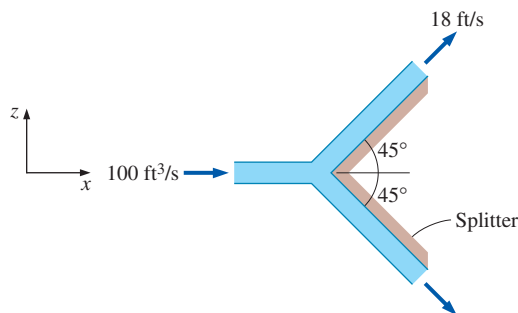



FIGURE P6-29E

**6-30E**  Reconsider Prob. 6-29E. Using EES (or other) software, investigate the effect of the splitter angle on the force exerted on the splitter in the incoming flow direction. Let the half splitter angle vary from  $0^\circ$  to  $180^\circ$  in increments of  $10^\circ$ . Tabulate and plot your results, and draw some conclusions.

**6-31** A horizontal  $5\text{-cm}$ -diameter water jet with a velocity of  $18 \text{ m/s}$  impinges normally upon a vertical plate of mass  $1000 \text{ kg}$ . The plate rides on a nearly frictionless track and is initially stationary. When the jet strikes the plate, the plate begins to move in the direction of the jet. The water always

splatters in the plane of the retreating plate. Determine (a) the acceleration of the plate when the jet first strikes it (time = 0), (b) the time it takes for the plate to reach a velocity of  $9 \text{ m/s}$ , and (c) the plate velocity  $20 \text{ s}$  after the jet first strikes the plate. For simplicity, assume the velocity of the jet is increased as the cart moves such that the impulse force exerted by the water jet on the plate remains constant.

**6-32E** A fan with  $24\text{-in}$ -diameter blades moves  $2000 \text{ cfm}$  (cubic feet per minute) of air at  $70^\circ\text{F}$  at sea level. Determine (a) the force required to hold the fan and (b) the minimum power input required for the fan. Choose a control volume sufficiently large to contain the fan, with the inlet sufficiently far upstream so that the gage pressure at the inlet is nearly zero. Assume air approaches the fan through a large area with negligible velocity and air exits the fan with a uniform velocity at atmospheric pressure through an imaginary cylinder whose diameter is the fan blade diameter. Answers: (a)  $0.820 \text{ lbf}$ , (b)  $5.91 \text{ W}$


**6-33E** A  $3\text{-in}$ -diameter horizontal jet of water, with velocity  $140 \text{ ft/s}$ , strikes a bent plate, which deflects the water by  $135^\circ$  from its original direction. How much force is required to hold the plate against the water stream and what is its direction? Disregard frictional and gravitational effects.

**6-34** Firefighters are holding a nozzle at the end of a hose while trying to extinguish a fire. If the nozzle exit diameter is  $8 \text{ cm}$  and the water flow rate is  $12 \text{ m}^3/\text{min}$ , determine (a) the average water exit velocity and (b) the horizontal resistance force required of the firefighters to hold the nozzle. Answers: (a)  $39.8 \text{ m/s}$ , (b)  $7958 \text{ N}$



FIGURE P6-34

**6-35** A  $5\text{-cm}$ -diameter horizontal jet of water with a velocity of  $40 \text{ m/s}$  relative to the ground strikes a flat plate that is moving in the same direction as the jet at a velocity of  $10 \text{ m/s}$ . The water splatters in all directions in the plane of the plate. How much force does the water stream exert on the plate?

**6-36**  Reconsider Prob. 6-35. Using EES (or other) software, investigate the effect of the plate velocity on the force exerted on the plate. Let the plate velocity vary from  $0$  to  $30 \text{ m/s}$ , in increments of  $3 \text{ m/s}$ . Tabulate and plot your results.

**6-37E** A 3-in-diameter horizontal water jet having a velocity of 90 ft/s strikes a curved plate, which deflects the water  $180^\circ$  at the same speed. Ignoring the frictional effects, determine the force required to hold the plate against the water stream.

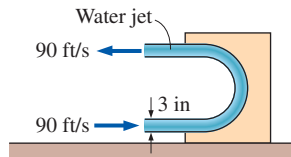


FIGURE P6-37

**6-38** An unloaded helicopter of mass 12,000 kg hovers at sea level while it is being loaded. In the unloaded hover mode, the blades rotate at 550 rpm. The horizontal blades above the helicopter cause a 18-m-diameter air mass to move downward at an average velocity proportional to the overhead blade rotational velocity (rpm). A load of 14,000 kg is loaded onto the helicopter, and the helicopter slowly rises. Determine (a) the volumetric airflow rate downdraft that the helicopter generates during unloaded hover and the required power input and (b) the rpm of the helicopter blades to hover with the 14,000-kg load and the required power input. Take the density of atmospheric air to be  $1.18 \text{ kg/m}^3$ . Assume air approaches the blades from the top through a large area with negligible velocity and air is forced by the blades to move down with a uniform velocity through an imaginary cylinder whose base is the blade span area.

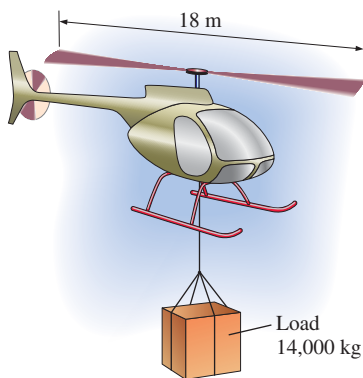


FIGURE P6-38

**6-39** Reconsider the helicopter in Prob. 6-38, except that it is hovering on top of a 2800-m-high mountain where the air density is  $0.928 \text{ kg/m}^3$ . Noting that the unloaded helicopter blades must rotate at 550 rpm to hover at sea level, determine the blade rotational velocity to hover at the higher altitude. Also determine the percent increase in the required power input to hover at 3000-m altitude relative to that at sea level. *Answers: 620 rpm, 12.8 percent*

**6-40** Water is flowing through a 10-cm-diameter water pipe at a rate of  $0.1 \text{ m}^3/\text{s}$ . Now a diffuser with an outlet diameter of 20 cm is bolted to the pipe in order to slow down water, as shown in Fig. P6-40. Disregarding frictional effects, determine the force exerted on the bolts due to the water flow.

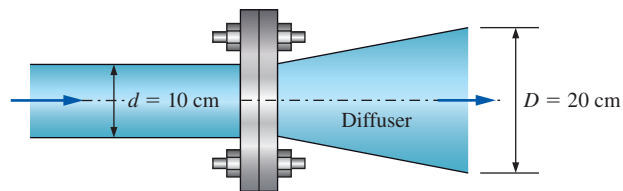


FIGURE P6-40

**6-41** The weight of a water tank open to the atmosphere is balanced by a counterweight, as shown in Fig. P6-41. There is a 4-cm hole at the bottom of the tank with a discharge coefficient of 0.90, and water level in the tank is maintained constant at 50 cm by water entering the tank horizontally. Determine how much mass must be added to or removed from the counterweight to maintain balance when the hole at the bottom is opened.

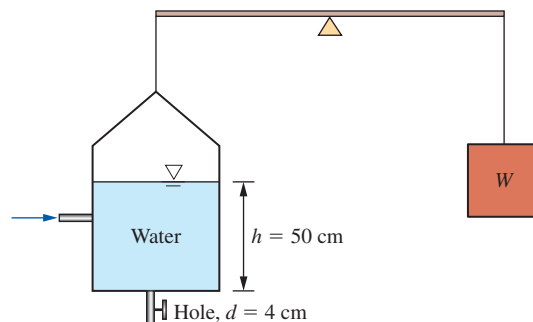



FIGURE P6-41

**6-42**  Commercially available large wind turbines have blade span diameters larger than 100 m and generate over 3 MW of electric power at peak design conditions.

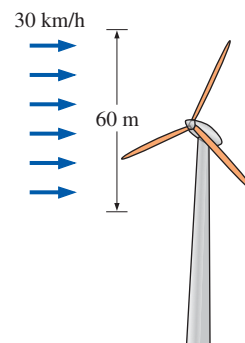


FIGURE P6-42

Consider a wind turbine with a 60-m blade span subjected to 30-km/h steady winds. If the combined turbine-generator efficiency of the wind turbine is 32 percent, determine (a) the power generated by the turbine and (b) the horizontal force exerted by the wind on the supporting mast of the turbine. Take the density of air to be  $1.25 \text{ kg/m}^3$ , and disregard frictional effects on mast.

**6-43** Water enters a centrifugal pump axially at atmospheric pressure at a rate of  $0.09 \text{ m}^3/\text{s}$  and at a velocity of  $5 \text{ m/s}$ , and leaves in the normal direction along the pump casing, as shown in Fig. P6-43. Determine the force acting on the shaft (which is also the force acting on the bearing of the shaft) in the axial direction.

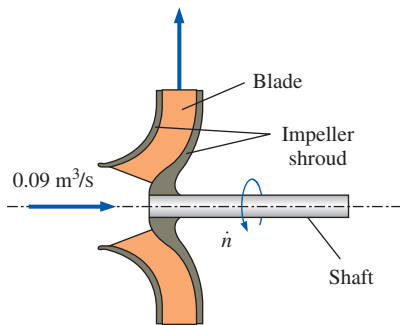


FIGURE P6-43

**6-44** An incompressible fluid of density  $\rho$  and viscosity  $\mu$  flows through a curved duct that turns the flow  $180^\circ$ . The duct cross-sectional area remains constant. The average velocity, momentum flux correction factor, and gage pressure are known at the inlet (1) and outlet (2), as in Fig. P6-44. (a) Write an expression for the horizontal force  $F_x$  of the fluid on the walls of the duct in terms of the given variables. (b) Verify your expression by plugging in the following values:  $\rho = 998.2 \text{ kg/m}^3$ ,  $\mu = 1.003 \times 10^{-3} \text{ kg/m}\cdot\text{s}$ ,  $A_1 = A_2 = 0.025 \text{ m}^2$ ,  $\beta_1 = 1.01$ ,  $\beta_2 = 1.03$ ,  $V_1 = 10 \text{ m/s}$ ,  $P_{1,\text{gage}} = 78.47 \text{ kPa}$ , and  $P_{2,\text{gage}} = 65.23 \text{ kPa}$ . *Answer: (b)  $F_x = 8680 \text{ N}$  to the right*

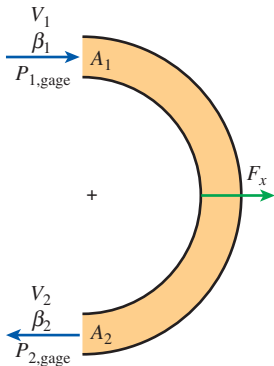


FIGURE P6-44

**6-45** Consider the curved duct of Prob. 6-44, except allow the cross-sectional area to vary along the duct ( $A_1 \neq A_2$ ). (a) Write an expression for the horizontal force  $F_x$  of the fluid on the walls of the duct in terms of the given variables. (b) Verify your expression by plugging in the following values:  $\rho = 998.2 \text{ kg/m}^3$ ,  $A_1 = 0.025 \text{ m}^2$ ,  $A_2 = 0.015 \text{ m}^2$ ,  $\beta_1 = 1.02$ ,  $\beta_2 = 1.04$ ,  $V_1 = 20 \text{ m/s}$ ,  $P_{1,\text{gage}} = 88.34 \text{ kPa}$ , and  $P_{2,\text{gage}} = 67.48 \text{ kPa}$ . *Answer: (b)  $F_x = 30,700 \text{ N}$  to the right*

**6-46** As a follow-up to Prob. 6-44, it turns out that for a large enough area ratio  $A_2/A_1$ , the inlet pressure is actually *smaller* than the outlet pressure! Explain how this can be true in light of the fact that there is friction and other irreversibilities due to turbulence, and pressure must be lost along the axis of the duct to overcome these irreversibilities.

**6-47** An incompressible fluid of density  $\rho$  and viscosity  $\mu$  flows through a curved duct that turns the flow through angle  $\theta$ . The cross-sectional area also changes. The average velocity, momentum flux correction factor, gage pressure, and area are known at the inlet (1) and outlet (2), as in Fig. P6-47. (a) Write an expression for the horizontal force  $F_x$  of the fluid on the walls of the duct in terms of the given variables. (b) Verify your expression by plugging in the following values:  $\theta = 135^\circ$ ,  $\rho = 998.2 \text{ kg/m}^3$ ,  $\mu = 1.003 \times 10^{-3} \text{ kg/m}\cdot\text{s}$ ,  $A_1 = 0.025 \text{ m}^2$ ,  $A_2 = 0.050 \text{ m}^2$ ,  $\beta_1 = 1.01$ ,  $\beta_2 = 1.03$ ,  $V_1 = 6 \text{ m/s}$ ,  $P_{1,\text{gage}} = 78.47 \text{ kPa}$ , and  $P_{2,\text{gage}} = 65.23 \text{ kPa}$ . (*Hint: You will first need to solve for  $V_2$ .*) (c) At what turning angle is the force maximized? *Answers: (b)  $F_x = 5500 \text{ N}$  to the right, (c)  $180^\circ$*

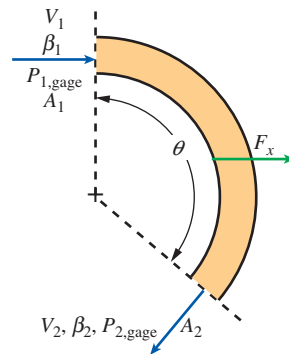


FIGURE P6-47

**6-48** Water of density  $\rho = 998.2 \text{ kg/m}^3$  flows through a fireman's nozzle—a converging section of pipe that accelerates the flow. The inlet diameter is  $d_1 = 0.100 \text{ m}$ , and the outlet diameter is  $d_2 = 0.050 \text{ m}$ . The average velocity, momentum flux correction factor, and gage pressure are known at the inlet (1) and outlet (2), as in Fig. P6-48. (a) Write an expression for the horizontal force  $F_x$  of the fluid on the walls of the nozzle in terms of the given variables. (b) Verify your expression by plugging in the following values:  $\beta_1 = 1.03$ ,  $\beta_2 = 1.02$ ,  $V_1 = 4 \text{ m/s}$ ,  $P_{1,\text{gage}} = 123,000 \text{ Pa}$ , and  $P_{2,\text{gage}} = 0 \text{ Pa}$ . *Answer: (b)  $F_x = 583 \text{ N}$  to the right*

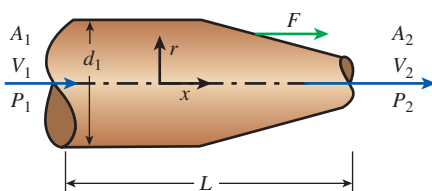


FIGURE P6-48

**6-49** Water flowing in a horizontal 25-cm-diameter pipe at 8 m/s and 300 kPa gage enters a 90° bend reducing section, which connects to a 15-cm-diameter vertical pipe. The inlet of the bend is 50 cm above the exit. Neglecting any frictional and gravitational effects, determine the net resultant force exerted on the reducer by the water. Take the momentum-flux correction factor to be 1.04.

**6-50** A sluice gate, which controls flow rate in a channel by simply raising or lowering a vertical plate, is commonly used in irrigation systems. A force is exerted on the gate due to the difference between the water heights  $y_1$  and  $y_2$  and the flow velocities  $V_1$  and  $V_2$  upstream and downstream from the gate, respectively. Take the width of the sluice gate (into the page) to be  $w$ . Wall shear stresses along the channel walls may be ignored, and for simplicity, we assume steady, uniform flow at locations 1 and 2. Develop a relationship for the force  $F_R$  acting on the sluice gate as a function of depths  $y_1$  and  $y_2$ , mass flow rate  $\dot{m}$ , gravitational constant  $g$ , gate width  $w$ , and water density  $\rho$ .

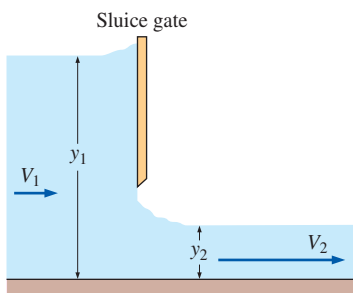


FIGURE P6-50

### Angular Momentum Equation

**6-51C** How is the angular momentum equation obtained from Reynolds transport equations?

**6-52C** Express the angular momentum equation in scalar form about a specified axis of rotation for a fixed control volume for steady and uniform flow.

**6-53C** Express the unsteady angular momentum equation in vector form for a control volume that has a constant moment of inertia  $I$ , no external moments applied, one outgoing uniform flow stream of velocity  $\vec{V}$ , and mass flow rate  $\dot{m}$ .

**6-54C** Consider two rigid bodies having the same mass and angular speed. Do you think these two bodies must have the same angular momentum? Explain.

**6-55** Water is flowing through a 15-cm-diameter pipe that consists of a 3-m-long vertical and 2-m-long horizontal section with a 90° elbow at the exit to force the water to be discharged downward, as shown in Fig. P6-55, in the vertical direction. Water discharges to atmospheric air at a velocity of 7 m/s, and the mass of the pipe section when filled with water is 15 kg per meter length. Determine the moment acting at the intersection of the vertical and horizontal sections of the pipe (point A). What would your answer be if the flow were discharged upward instead of downward?

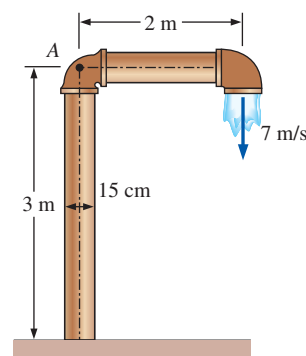


FIGURE P6-55

**6-56E** A large lawn sprinkler with two identical arms is used to generate electric power by attaching a generator to its rotating head. Water enters the sprinkler from the base along the axis of rotation at a rate of 5 gal/s and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 180 rpm in a horizontal plane. The diameter of each jet is 0.5 in, and the normal distance between the axis of rotation and the center of each nozzle is 2 ft. Determine the maximum possible electrical power produced.

**6-57E** Reconsider the lawn sprinkler in Prob. 6-56E. If the rotating head is somehow stuck, determine the moment acting on the head.

**6-58** The impeller of a centrifugal pump has inner and outer diameters of 13 and 30 cm, respectively, and a flow rate of 0.15 m³/s at a rotational speed of 1200 rpm. The blade width of the impeller is 8 cm at the inlet and 3.5 cm at the outlet. If water enters the impeller in the radial direction and exits at an angle of 60° from the radial direction, determine the minimum power requirement for the pump.

**6-59** The impeller of a centrifugal blower has a radius of 18 cm and a blade width of 6.1 cm at the inlet, and a radius of 30 cm and a blade width of 3.4 cm at the outlet. The blower delivers atmospheric air at 20°C and 95 kPa. Disregarding any losses and assuming the tangential components of air



velocity at the inlet and the outlet to be equal to the impeller velocity at respective locations, determine the volumetric flow rate of air when the rotational speed of the shaft is 900 rpm and the power consumption of the blower is 120 W. Also determine the normal components of velocity at the inlet and outlet of the impeller.

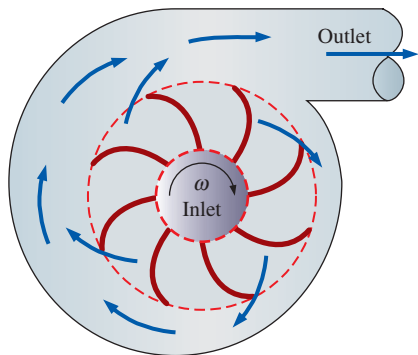


FIGURE P6-59

**6-60** Water enters vertically and steadily at a rate of 35 L/s into the sprinkler shown in Fig. P6-60 with unequal arms and unequal discharge areas. The smaller jet has a discharge area of 3 cm<sup>2</sup> and a normal distance of 50 cm from the axis of rotation. The larger jet has a discharge area of 5 cm<sup>2</sup> and a normal distance of 35 cm from the axis of rotation. Disregarding any frictional effects, determine (a) the rotational speed of the sprinkler in rpm and (b) the torque required to prevent the sprinkler from rotating.

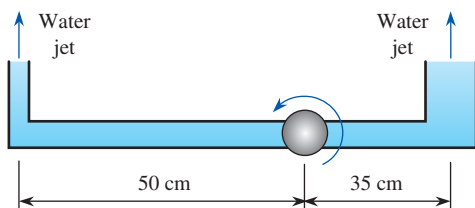


FIGURE P6-60

**6-61** Repeat Prob. 6-60 for a water flow rate of 50 L/s.

**6-62** Consider a centrifugal blower that has a radius of 20 cm and a blade width of 8.2 cm at the impeller inlet, and a radius of 45 cm and a blade width of 5.6 cm at the outlet. The blower delivers air at a rate of 0.70 m<sup>3</sup>/s at a rotational speed of 700 rpm. Assuming the air to enter the impeller in the radial direction and to exit at an angle of 50° from the radial direction, determine the minimum power consumption of the blower. Take the density of air to be 1.25 kg/m<sup>3</sup>.

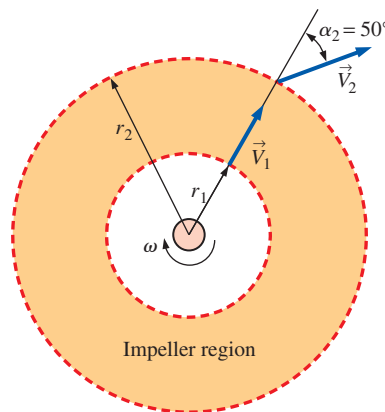



FIGURE P6-62

**6-63**  Reconsider Prob. 6-62. For the specified flow rate, investigate the effect of discharge angle  $\alpha_2$  on the minimum power input requirements. Assume the air to enter the impeller in the radial direction ( $\alpha_1 = 0^\circ$ ), and vary  $\alpha_2$  from 0° to 85° in increments of 5°. Plot the variation of power input versus  $\alpha_2$ , and discuss your results.

**6-64E** Water enters the impeller of a centrifugal pump radially at a rate of 45 cfm (cubic feet per minute) when the shaft is rotating at 500 rpm. The tangential component of absolute velocity of water at the exit of the 2-ft outer diameter impeller is 110 ft/s. Determine the torque applied to the impeller and the minimum power input to the pump.

*Answers: 160 lbf·ft, 11.3 kW*

**6-65** A lawn sprinkler with three identical arms is used to water a garden by rotating in a horizontal plane by the impulse caused by water flow. Water enters the sprinkler along the axis of rotation at a rate of 60 L/s and leaves the 1.5-cm-diameter nozzles in the tangential direction. The bearing applies a retarding torque of  $T_0 = 50 \text{ N}\cdot\text{m}$  due to friction at the anticipated operating speeds. For a normal distance of 40 cm between the axis of rotation and the center of the nozzles, determine the angular velocity of the sprinkler shaft.

**6-66** Pelton wheel turbines are commonly used in hydroelectric power plants to generate electric power. In these turbines, a high-speed jet at a velocity of  $V_j$  impinges on buckets, forcing the wheel to rotate. The buckets reverse the direction of the jet, and the jet leaves the bucket making an angle  $\beta$  with the direction of the jet, as shown in Fig. P6-66. Show that the power produced by a Pelton wheel of radius  $r$  rotating steadily at an angular velocity of  $\omega$  is  $\dot{W}_{\text{shaft}} = \rho \omega r \dot{V} (V_j - \omega r)(1 - \cos \beta)$ , where  $\rho$  is the density and  $\dot{V}$  is the volume flow rate of the fluid. Obtain the numerical value for  $\rho = 1000 \text{ kg/m}^3$ ,  $r = 2 \text{ m}$ ,  $\dot{V} = 10 \text{ m}^3/\text{s}$ ,  $\dot{n} = 150 \text{ rpm}$ ,  $\beta = 160^\circ$ , and  $V_j = 50 \text{ m/s}$ .



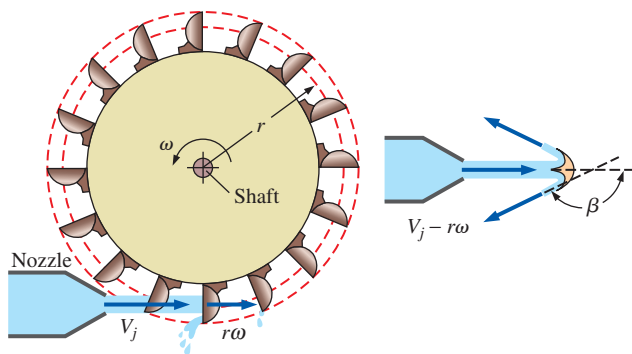



FIGURE P6-66

**6-67**  Reconsider Prob. 6-66. The maximum efficiency of the turbine occurs when  $\beta = 180^\circ$ , but this is not practical. Investigate the effect of  $\beta$  on the power generation by allowing it to vary from  $0^\circ$  to  $180^\circ$ . Do you think we are wasting a large fraction of power by using buckets with a  $\beta$  of  $160^\circ$ ?

### Review Problems

**6-68** Water flowing steadily at a rate of  $0.16 \text{ m}^3/\text{s}$  is deflected downward by an angled elbow as shown in Fig. P6-68. For  $D = 30 \text{ cm}$ ,  $d = 10 \text{ cm}$ , and  $h = 50 \text{ cm}$ , determine the force acting on the flanges of the elbow and the angle its line of action makes with the horizontal. Take the internal volume of the elbow to be  $0.03 \text{ m}^3$  and disregard the weight of the elbow material and the frictional effects.

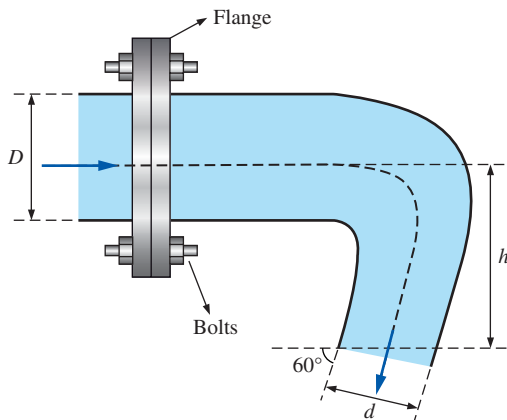


FIGURE P6-68

**6-69** Repeat Prob. 6-68 by taking into consideration the weight of the elbow whose mass is  $5 \text{ kg}$ .

**6-70** A  $12\text{-cm}$  diameter horizontal water jet with a speed of  $V_j = 25 \text{ m/s}$  relative to the ground is deflected by a  $40^\circ$  cone

moving to the left at  $V_c = 10 \text{ m/s}$ . Determine the external force,  $F$ , needed to maintain the motion of the cone. Disregard the gravity and surface shear effects and assume the cross-sectional area of water jet normal to the direction of motion remains constant throughout the flow. *Answer: 3240 N*

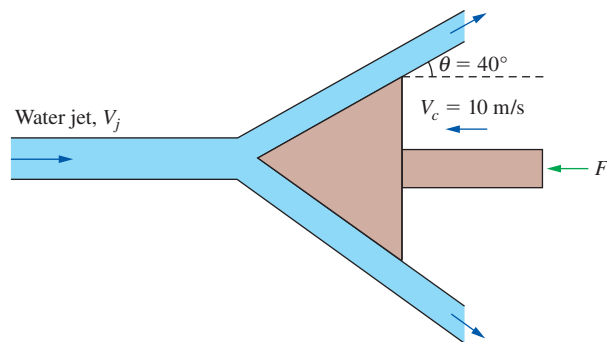


FIGURE P6-70

**6-71** Water enters vertically and steadily at a rate of  $10 \text{ L/s}$  into the sprinkler shown in Fig. P6-71. Both water jets have a diameter of  $1.2 \text{ cm}$ . Disregarding any frictional effects, determine (a) the rotational speed of the sprinkler in rpm and (b) the torque required to prevent the sprinkler from rotating.

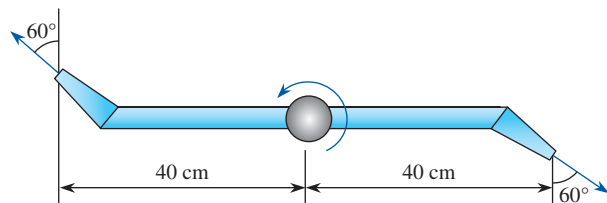


FIGURE P6-71

**6-72** Repeat Prob. 6-71 for the case of unequal arms—the left one being  $60 \text{ cm}$  and the right one  $20 \text{ cm}$  from the axis of rotation.

**6-73** A  $6\text{-cm}$ -diameter horizontal water jet having a velocity of  $25 \text{ m/s}$  strikes a vertical stationary flat plate. The water splatters in all directions in the plane of the plate. How much force is required to hold the plate against the water stream? *Answers: 1770 N*

**6-74** Consider steady developing laminar flow of water in a constant-diameter horizontal discharge pipe attached to a tank. The fluid enters the pipe with nearly uniform velocity  $V$  and pressure  $P_1$ . The velocity profile becomes parabolic after a certain distance with a momentum correction factor

of 2 while the pressure drops to  $P_2$ . Obtain a relation for the horizontal force acting on the bolts that hold the pipe attached to the tank.

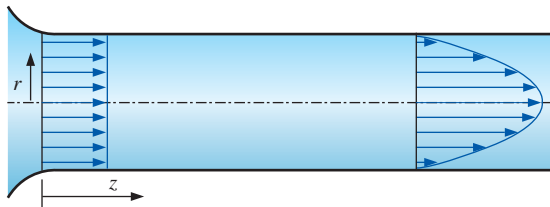


FIGURE P6-74

**6-75** A tripod holding a nozzle, which directs a 5-cm-diameter stream of water from a hose, is shown in Fig. P6-75. The nozzle mass is 10 kg when filled with water. The tripod is rated to provide 1800 N of holding force. A firefighter was standing 60 cm behind the nozzle and was hit by the nozzle when the tripod suddenly failed and released the nozzle. You have been hired as an accident reconstructionist and, after testing the tripod, have determined that as water flow rate increased, it did collapse at 1800 N. In your final report you must state the water velocity and the flow rate consistent with the failure and the nozzle velocity when it hit the firefighter. For simplicity, ignore pressure and momentum effects in the upstream portion of the hose. *Answers:* 30.3 m/s, 0.0595 m<sup>3</sup>/s, 14.7 m/s

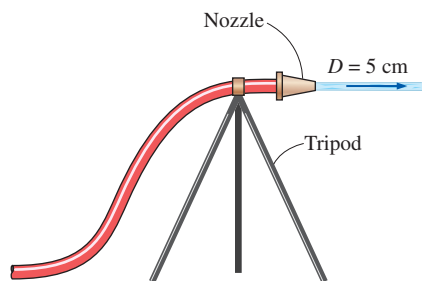


FIGURE P6-75

**6-76** Consider an airplane with a jet engine attached to the tail section that expels combustion gases at a rate of 18 kg/s with a velocity of  $V = 300$  m/s relative to the plane. During landing, a thrust reverser (which serves as a brake for the aircraft and facilitates landing on a short runway) is lowered in the path of the exhaust jet, which deflects the exhaust from rearward to  $150^\circ$ . Determine (a) the thrust (forward force) that the engine produces prior to the insertion of the thrust reverser and (b) the braking force produced after the thrust reverser is deployed.

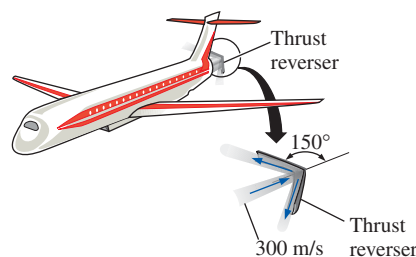



FIGURE P6-76

**6-77**  Reconsider Prob. 6-76. Using EES (or other) software, investigate the effect of thrust reverser angle on the braking force exerted on the airplane. Let the reverser angle vary from  $0^\circ$  (no reversing) to  $180^\circ$  (full reversing) in increments of  $10^\circ$ . Tabulate and plot your results and draw conclusions.

**6-78E** A spacecraft cruising in space at a constant velocity of 2000 ft/s has a mass of 25,000 lbm. To slow down the spacecraft, a solid fuel rocket is fired, and the combustion gases leave the rocket at a constant rate of 150 lbm/s at a velocity of 5000 ft/s in the same direction as the spacecraft for a period of 5 s. Assuming the mass of the spacecraft remains constant, determine (a) the deceleration of the spacecraft during this 5-s period, (b) the change of velocity of the spacecraft during this time period, and (c) the thrust exerted on the spacecraft.

**6-79** A 60-kg ice skater is standing on ice with ice skates (negligible friction). She is holding a flexible hose (essentially weightless) that directs a 2-cm-diameter stream of water horizontally parallel to her skates. The water velocity at the hose outlet is 10 m/s relative to the skater. If she is initially standing still, determine (a) the velocity of the skater and the distance she travels in 5 s and (b) how long it will take to move 5 m and the velocity at that moment. *Answers:* (a) 2.62 m/s, 6.54 m, (b) 4.4 s, 2.3 m/s



FIGURE P6-79

**6-80** A 5-cm-diameter horizontal jet of water, with velocity 30 m/s, strikes the tip of a horizontal cone, which deflects the water by  $45^\circ$  from its original direction. How much force is required to hold the cone against the water stream?

**6-81** Water is flowing into and discharging from a pipe U-section as shown in Fig. P6-81. At flange (1), the total absolute pressure is 200 kPa, and 55 kg/s flows into the pipe. At flange (2), the total pressure is 150 kPa. At location (3), 15 kg/s of water discharges to the atmosphere, which is at 100 kPa. Determine the total  $x$ - and  $z$ -forces at the two flanges connecting the pipe. Discuss the significance of gravity force for this problem. Take the momentum-flux correction factor to be 1.03 throughout the pipes.

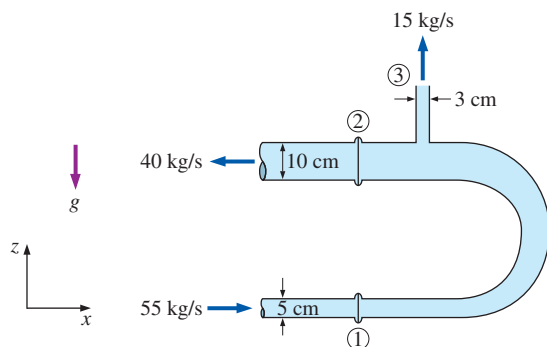


FIGURE P6-81

**6-82** Indiana Jones needs to ascend a 10-m-high building. There is a large hose filled with pressurized water hanging

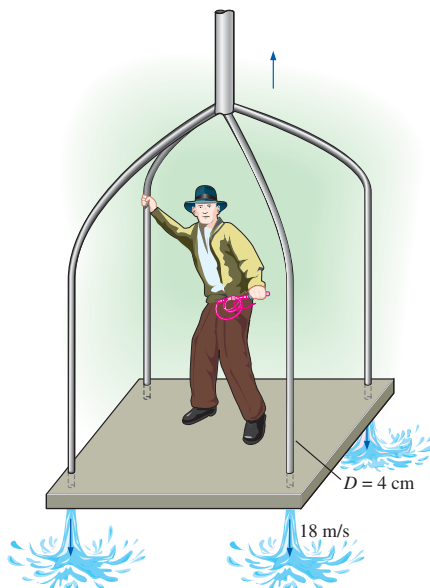


FIGURE P6-82

down from the building top. He builds a square platform and mounts four 4-cm-diameter nozzles pointing down at each corner. By connecting hose branches, a water jet with 15-m/s velocity can be produced from each nozzle. Jones, the platform, and the nozzles have a combined mass of 150 kg. Determine (a) the minimum water jet velocity needed to raise the system, (b) how long it takes for the system to rise 10 m when the water jet velocity is 18 m/s and the velocity of the platform at that moment, and (c) how much higher will the momentum raise Jones if he shuts off the water at the moment the platform reaches 10 m above the ground. How much time does he have to jump from the platform to the roof? *Answers: (a) 17.1 m/s, (b) 4.37 s, 4.57 m/s, (c) 1.07 m, 0.933 s*

**6-83E** An engineering student considers using a fan as a levitation demonstration. She plans to face the box-enclosed fan so the air blast is directed face down through a 3-ft-diameter blade span area. The system weighs 5 lbf, and the student will secure the system from rotating. By increasing the power to the fan, she plans to increase the blade rpm and air exit velocity until the exhaust provides sufficient upward force to cause the box fan to hover in the air. Determine (a) the air exit velocity to produce 5 lbf, (b) the volumetric flow rate needed, and (c) the minimum mechanical power that must be supplied to the airstream. Take the air density to be  $0.078 \text{ lbm/ft}^3$ .

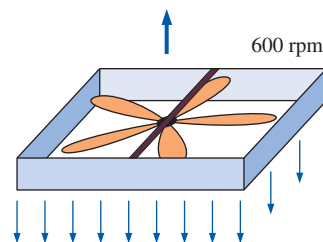


FIGURE P6-83E

**6-84** Nearly frictionless vertical guide rails maintain a plate of mass  $m_p$  in a horizontal position, such that it can slide freely in the vertical direction. A nozzle directs a water stream of area  $A$  against the plate underside. The water jet splatters in the plate plane, applying an upward force against the plate. The water flow rate  $\dot{m}$  (kg/s) can be controlled. Assume that distances are short, so the velocity of the rising jet can be considered constant with height. (a) Determine the minimum mass flow rate  $\dot{m}_{\min}$  necessary to just levitate the plate and obtain a relation for the steady-state velocity of the upward moving plate for  $\dot{m} > \dot{m}_{\min}$ . (b) At time  $t = 0$ , the plate is at rest, and the water jet with  $\dot{m} > \dot{m}_{\min}$  is suddenly turned on. Apply a force balance to the plate and obtain the integral that relates velocity to time (do not solve).

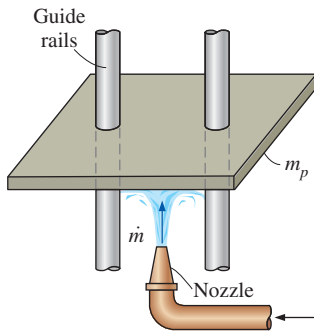


FIGURE P6–84

**6–85** A walnut with a mass of 50 g requires a force of 200 N applied continuously for 0.002 s to be cracked. If walnuts are to be cracked by dropping them from a high place onto a hard surface, determine the minimum height required. Disregard air friction.

**6–86** A 7-cm diameter vertical water jet is injected upwards by a nozzle at a speed of 15 m/s. Determine the maximum weight of a flat plate that can be supported by this water jet at a height of 2 m from the nozzle.

**6–87** Repeat Prob. 6–86 for a height of 8 m from the nozzle.

**6–88** Show that the force exerted by a liquid jet on a stationary nozzle as it leaves with a velocity  $V$  is proportional to  $V^2$  or, alternatively, to  $\dot{m}^2$ . Assume the jet stream is perpendicular to the incoming liquid flow line.

**6–89** A soldier jumps from a plane and opens his parachute when his velocity reaches the terminal velocity  $V_T$ . The parachute slows him down to his landing velocity of  $V_F$ . After the parachute is deployed, the air resistance is proportional to the velocity squared (i.e.,  $F = kV^2$ ). The soldier, his parachute, and his gear have a total mass of  $m$ . Show that  $k = mg/V_F^2$



FIGURE P6–89

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and develop a relation for the soldier's velocity after he opens the parachute at time  $t = 0$ .

$$\text{Answer: } V = V_F \frac{V_T + V_F + (V_T - V_F)e^{-2gt/V_F}}{V_T + V_F - (V_T - V_F)e^{-2gt/V_F}}$$

**6–90** A horizontal water jet with a flow rate of  $\dot{V}$  and cross-sectional area of  $A$  drives a covered cart of mass  $m_c$  along a level and nearly frictionless path. The jet enters a hole at the rear of the cart and all water that enters the cart is retained, increasing the system mass. The relative velocity between the jet of constant velocity  $V_J$  and the cart of variable velocity  $V$  is  $V_J - V$ . If the cart is initially empty and stationary when the jet action is initiated, develop a relation (integral form is acceptable) for cart velocity versus time.

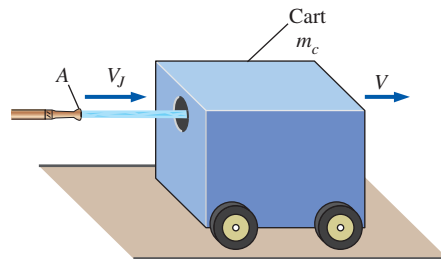


FIGURE P6–90

**6–91** Water accelerated by a nozzle enters the impeller of a turbine through its outer edge of diameter  $D$  with a velocity of  $V$  making an angle  $\alpha$  with the radial direction at a mass flow rate of  $\dot{m}$ . Water leaves the impeller in the radial direction. If the angular speed of the turbine shaft is  $\dot{n}$ , show that the maximum power that can be generated by this radial turbine is  $\dot{W}_{\text{shaft}} = \pi \dot{n} \dot{m} D V \sin \alpha$ .

**6–92** Water enters a two-armed lawn sprinkler along the vertical axis at a rate of 75 L/s, and leaves the sprinkler nozzles as 2-cm diameter jets at an angle of  $\theta$  from the tangential direction, as shown in Fig. P6–92. The length of each sprinkler

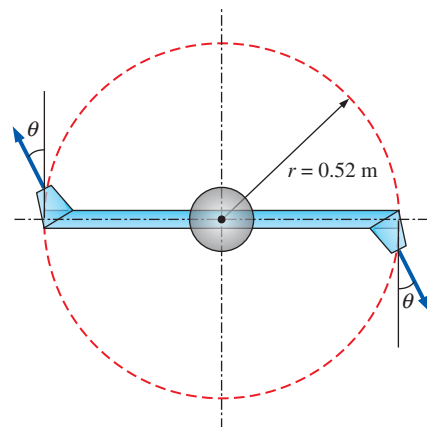



FIGURE P6–92

arm is 0.52 m. Disregarding any frictional effects, determine the rate of rotation  $\dot{n}$  of the sprinkler in rev/min for (a)  $\theta = 0^\circ$ , (b)  $\theta = 30^\circ$ , and (c)  $\theta = 60^\circ$ .

**6-93**  Reconsider Prob. 6-92. For the specified flow rate, investigate the effect of discharge angle  $\theta$  on the rate of rotation  $\dot{n}$  by varying  $\theta$  from  $0^\circ$  to  $90^\circ$  in increments of  $10^\circ$ . Plot the rate of rotation versus  $\theta$ , and discuss your results.

**6-94** A stationary water tank of diameter  $D$  is mounted on wheels and is placed on a nearly frictionless level surface. A smooth hole of diameter  $D_o$  near the bottom of the tank allows water to jet horizontally and rearward and the water jet force propels the system forward. The water in the tank is much heavier than the tank-and-wheel assembly, so only the mass of water remaining in the tank needs to be considered in this problem. Considering the decrease in the mass of water with time, develop relations for (a) the acceleration, (b) the velocity, and (c) the distance traveled by the system as a function of time.

**6-95** An orbiting satellite has a mass of 3400 kg and is traveling at a constant velocity of  $V_0$ . To alter its orbit, an attached rocket discharges 100 kg of gases from the reaction of solid fuel at a speed of 3000 m/s relative to the satellite in a direction opposite  $V_0$ . The fuel discharge rate is constant for 3 s. Determine (a) the thrust exerted on the satellite, (b) the acceleration of the satellite during this 3-s period, and (c) the change of velocity of the satellite during this time period.

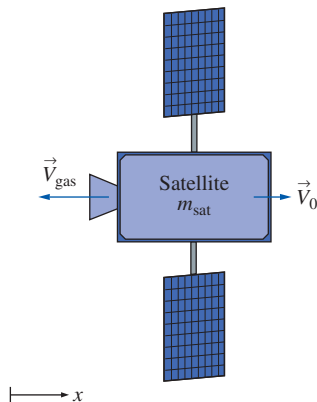


FIGURE P6-95

**6-96** Water enters a mixed flow pump axially at a rate of  $0.3 \text{ m}^3/\text{s}$  and at a velocity of 7 m/s, and is discharged to the atmosphere at an angle of  $75^\circ$  from the horizontal, as shown in Fig. P6-96. If the discharge flow area is half the inlet area, determine the force acting on the shaft in the axial direction.

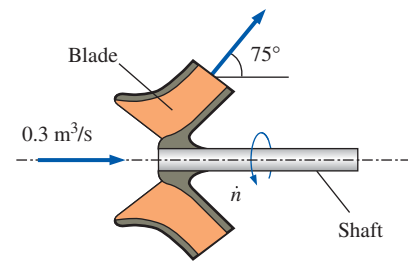


FIGURE P6-96

**6-97** Water flows steadily through a splitter as shown in Fig. P6-97 with  $\dot{V}_1 = 0.08 \text{ m}^3/\text{s}$ ,  $\dot{V}_2 = 0.05 \text{ m}^3/\text{s}$ ,  $D_1 = D_2 = 12 \text{ cm}$ ,  $D_3 = 10 \text{ cm}$ . If the pressure readings at the inlet and outlets of the splitter are  $P_1 = 100 \text{ kPa}$ ,  $P_2 = 90 \text{ kPa}$  and  $P_3 = 80 \text{ kPa}$ , determine external force needed to hold the device fixed. Disregard the weight effects.

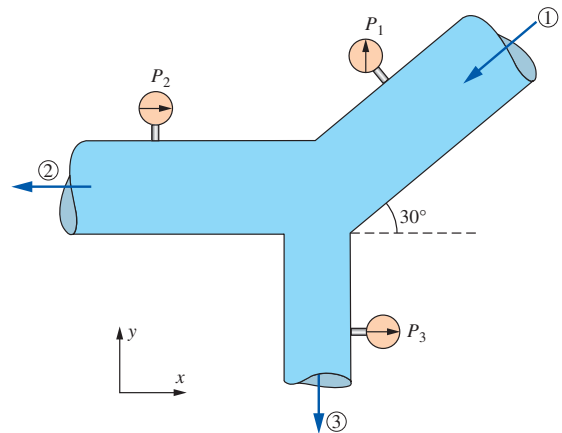


FIGURE P6-97

**6-98** Water is discharged from a pipe through a 1.2-m long 5-mm wide rectangular slit underneath of the pipe. Water discharge velocity profile is parabolic, varying from 3 m/s on one end of the slit to 7 m/s on the other, as shown in Fig. P6-98.

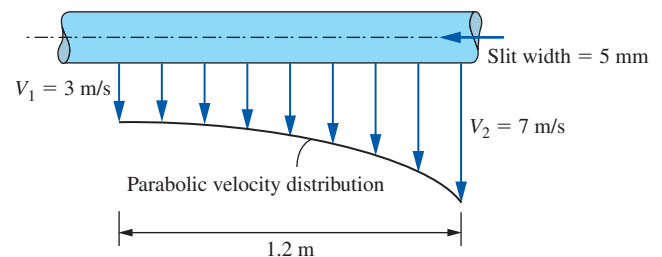


FIGURE P6-98



Determine (a) the rate of discharge through the slit and (b) the vertical force acting on the pipe due to this discharge process.

### Fundamentals of Engineering (FE) Exam Problems

**6-99** When determining the thrust developed by a jet engine, a wise choice of control volume is

- (a) Fixed control volume (b) Moving control volume  
(c) Deforming control volume (d) Moving or deforming control volume (e) None of these

**6-100** Consider an airplane cruising at 850 km/h to the right. If the velocity of exhaust gases is 700 km/h to the left relative to the ground, the velocity of the exhaust gases relative to the nozzle exit is

- (a) 1550 km/h (b) 850 km/h (c) 700 km/h  
(d) 350 km/h (e) 150 km/h

**6-101** Consider water flow through a horizontal, short garden hose at a rate of 30 kg/min. The velocity at the inlet is 1.5 m/s and that at the outlet is 14.5 m/s. Disregard the weight of the hose and water. Taking the momentum-flux correction factor to be 1.04 at both the inlet and the outlet, the anchoring force required to hold the hose in place is

- (a) 2.8 N (b) 8.6 N (c) 17.5 N (d) 27.9 N (e) 43.3 N

**6-102** Consider water flow through a horizontal, short garden hose at a rate of 30 kg/min. The velocity at the inlet is 1.5 m/s and that at the outlet is 11.5 m/s. The hose makes a 180° turn before the water is discharged. Disregard the weight of the hose and water. Taking the momentum-flux correction factor to be 1.04 at both the inlet and the outlet, the anchoring force required to hold the hose in place is

- (a) 7.6 N (b) 28.4 N (c) 16.6 N (d) 34.1 N  
(e) 11.9 N

**6-103** A water jet strikes a stationary vertical plate horizontally at a rate of 5 kg/s with a velocity of 35 km/h. Assume the water stream moves in the vertical direction after the strike. The force needed to prevent the plate from moving horizontally is

- (a) 15.5 N (b) 26.3 N (c) 19.7 N (d) 34.2 N (e) 48.6 N

**6-104** Consider water flow through a horizontal, short garden hose at a rate of 40 kg/min. The velocity at the inlet is 1.5 m/s and that at the outlet is 16 m/s. The hose makes a 90° turn to a vertical direction before the water is discharged. Disregard the weight of the hose and water. Taking the momentum-flux correction factor to be 1.04 at both the inlet and the outlet, the reaction force in the vertical direction required to hold the hose in place is

- (a) 11.1 N (b) 10.1 N (c) 9.3 N (d) 27.2 N (e) 28.9 N

**6-105** Consider water flow through a horizontal, short pipe at a rate of 80 kg/min. The velocity at the inlet is 1.5 m/s and that at the outlet is 16.5 m/s. The pipe makes a 90° turn to a vertical direction before the water is discharged. Disregard the weight of the pipe and water. Taking the momentum-flux correction factor to be 1.04 at both the inlet and the outlet,

the reaction force in the horizontal direction required to hold the pipe in place is

- (a) 73.7 N (b) 97.1 N (c) 99.2 N (d) 122 N (e) 153 N

**6-106** A water jet strikes a stationary horizontal plate vertically at a rate of 18 kg/s with a velocity of 24 m/s. The mass of the plate is 10 kg. Assume the water stream moves in the horizontal direction after the strike. The force needed to prevent the plate from moving vertically is

- (a) 192 N (b) 240 N (c) 334 N (d) 432 N (e) 530 N

**6-107** The velocity of wind at a wind turbine is measured to be 6 m/s. The blade span diameter is 24 m and the efficiency of the wind turbine is 29 percent. The density of air is 1.22 kg/m<sup>3</sup>. The horizontal force exerted by the wind on the supporting mast of the wind turbine is

- (a) 2524 N (b) 3127 N (c) 3475 N (d) 4138 N  
(e) 4313 N

**6-108** The velocity of wind at a wind turbine is measured to be 8 m/s. The blade span diameter is 12 m. The density of air is 1.2 kg/m<sup>3</sup>. If the horizontal force exerted by the wind on the supporting mast of the wind turbine is 1620 N, the efficiency of the wind turbine is

- (a) 27.5% (b) 31.7% (c) 29.5% (d) 35.1% (e) 33.8%

**6-109** The shaft of a turbine rotates at a speed of 800 rpm. If the torque of the shaft is 350 N·m, the shaft power is

- (a) 112 kW (b) 176 kW (c) 293 kW (d) 350 kW  
(e) 405 kW

**6-110** A 3-cm-diameter horizontal pipe attached to a surface makes a 90° turn to a vertical upward direction before the water is discharged at a velocity of 9 m/s. The horizontal section is 5 m long and the vertical section is 4 m long. Neglecting the mass of the water contained in the pipe, the bending moment acting on the base of the pipe on the wall is

- (a) 286 N·m (b) 229 N·m (c) 207 N·m  
(d) 175 N·m (e) 124 N·m

**6-111** A 3-cm-diameter horizontal pipe attached to a surface makes a 90° turn to a vertical upward direction before the water is discharged at a velocity of 6 m/s. The horizontal section is 5 m long and the vertical section is 4 m long. Neglecting the mass of the pipe and considering the weight of the water contained in the pipe, the bending moment acting on the base of the pipe on the wall is

- (a) 11.9 N·m (b) 46.7 N·m (c) 127 N·m  
(d) 104 N·m (e) 74.8 N·m

**6-112** A large lawn sprinkler with four identical arms is to be converted into a turbine to generate electric power by attaching a generator to its rotating head. Water enters the sprinkler from the base along the axis of rotation at a rate of 15 kg/s and leaves the nozzles in the tangential direction at a velocity of 50 m/s relative to the rotating nozzle. The sprinkler rotates at a rate of 400 rpm in a horizontal plane. The normal distance between the axis of rotation and the center of each nozzle is 30 cm. Estimate the electric power produced.



- (a) 5430 W (b) 6288 W (c) 6634 W (d) 7056 W  
(e) 7875 W

**6-113** Consider the impeller of a centrifugal pump with a rotational speed of 900 rpm and a flow rate of 95 kg/min. The impeller radii at the inlet and outlet are 7 cm and 16 cm, respectively. Assuming that the tangential fluid velocity is equal to the blade angular velocity both at the inlet and the exit, the power requirement of the pump is  
(a) 83 W (b) 291 W (c) 409 W (d) 756 W (e) 1125 W

**6-114** Water enters the impeller of a centrifugal pump radially at a rate of 450 L/min when the shaft is rotating

at 400 rpm. The tangential component of absolute velocity of water at the exit of the 70-cm outer diameter impeller is 55 m/s. The torque applied to the impeller is

- (a) 144 N·m (b) 93.6 N·m (c) 187 N·m  
(d) 112 N·m (e) 235 N·m

### Design and Essay Problem

**6-115** Visit a fire station and obtain information about flow rates through hoses and discharge diameters. Using this information, calculate the impulse force to which the firefighters are subjected when holding a fire hose.

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